LINEAR CONNEXIONS WITH ZERO TORSION AND RECURRENT CURVATURE

BY

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Introduction. Let M be a connected n-dimensional C^∞-manifold. If S is any tensor (field) on M, then

(i) $S(u)$ denotes the value of S at the point $u \in M$;

(ii) $S=0$ means that S is everywhere zero, i.e. $S(u)=0$ for every $u \in M$;

(iii) $S \not\equiv 0$ means that S is not everywhere zero, i.e. $S(u) \neq 0$ for some $u \in M$;

(iv) $S \neq 0$ means that S is nowhere zero, i.e. $S(u)=0$ for no $u \in M$.

Throughout this paper, a tensor of type (1, 0) will be called a *vector*, and a tensor of type (0, 1) a *covector*. Summation over a repeated index, Latin or Greek, is always implied.

Let γ be a linear connexion on M. If $\Gamma_{ji}{}^h$ $(1 \leqq a, h, i, j, \cdots, \leqq n)$ are the components of γ in the local coordinate system (U, u^h) in M, then the components in (U, u^h) of the torsion tensor T and the curvature tensor R on M are respectively

$$T_{ji}{}^h = \Gamma_{ji}{}^h - \Gamma_{ij}{}^h,$$

$$R_{kji}{}^h = \partial_k \Gamma_{ji}{}^h - \partial_j \Gamma_{ki}{}^h + \Gamma_{ka}{}^h \Gamma_{ji}{}^a - \Gamma_{ja}{}^h \Gamma_{ki}{}^a,$$

where $\partial_k = \partial/\partial u^k$. If γ has zero torsion, i.e., if $T=0$, then the curvature tensor R satisfies the two Bianchi identities

$$R_{kji}{}^h + R_{jik}{}^h + R_{ikj}{}^h = 0, \tag{0.1}$$

$$\nabla_l R_{kji}{}^h + \nabla_k R_{jli}{}^h + \nabla_j R_{lki}{}^h = 0, \tag{0.2}$$

where ∇ denotes covariant differentiation with respect to γ.

A linear connexion γ on M is said to be *with recurrent curvature* if its curvature tensor R is not everywhere zero and if the covariant derivative of R is equal to the tensor product of a covector and R itself. We express these conditions in symbol by $R \not\equiv 0$ and $\nabla R = W \otimes R$, and call W the *recurrence covector*. By Theorem 1.2, the conditions $R \not\equiv 0$ and $\nabla R = W \otimes R$ imply that $R \neq 0$. Hence for a linear connexion with recurrent curvature, the curvature tensor is nowhere zero, i.e. $R \neq 0$. In what follows, whenever the condition $\nabla R = W \otimes R$ is assumed, the condition $R \not\equiv 0$ and consequently also the condition $R \neq 0$ are always understood.

Recently, generalizing a result of K. Nomizu concerning linear connexions

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with $\nabla R=0$, the author obtained a geometrical condition, in terms of the basic and fundamental vector fields on the frame bundle over M, for a linear connexion on M to be with recurrent curvature (Wong [13, Theorem 4.2]). Except for this general result, linear connexions with recurrent curvature have so far been studied only in the following special cases:

1. Riemannian connexions[1] with $\nabla R=0$ (i.e. E. Cartan's symmetric spaces).

2. Linear connexions with $T=0$, $\nabla R=0$, or with $\nabla T=0$, $\nabla R=0$ (Nomizu [3; 4]).

3. Riemannian connexions[1] with $\nabla R=W\otimes R$, $W\neq 0$ (Ruse [5], Walker [8], Hlavaty [2]).

4. Certain type of linear connexions with $T=0$, $\nabla R=W\otimes R$ and $W\neq 0$ which include, except for a trivial case, all the Riemannian connexions in Case 3 above (Wong [9]).

5. Subflat linear connexions (i.e. linear connexions which admit locally $n-1$ parallel fields of vectors and $n-1$ parallel fields of covectors) with $T=0$, $\nabla R=W\otimes R$ and $W\neq 0$ (Wong [11]).

6. Projectively flat connexions with $T=0$, $\nabla R=W\otimes R$, and $W\neq 0$ or $W=0$ (Wong and Yano [12]).

Furthermore, in Cases 3–6, only local properties were considered.

In this paper, we shall study the more general case of linear connexions with $T=0$, $\nabla R=W\otimes R$, $W\not\equiv 0$, i.e. linear connexions with zero torsion and recurrent curvature for which the recurrence covector W is not everywhere zero. Both local and global properties are obtained. The method we use is a combination of the classical tensor calculus and a method of piecing together locally defined tensor fields or fields of planes into global tensor fields or fields of planes on M. On two occasions, we apply Steenrod's theorem on existence of cross-sections in fiber bundles. It appears that neither the index-free presentation nor the method of exterior forms lends itself to the study of problems of the type considered in this paper. We assume that $n>2$. The case $n=2$ requires special treatment and will be considered in another paper (Wong [14]).

§§1.1–1.3 are preliminary in nature. They contain a number of results concerning parallel fields of planes or coplanes on M, recurrent tensors, and certain decomposition of recurrent tensors. All of them will play an important part in our later work. In §§2.1–2.4, we give a few rather easy but indispensable results on linear connexions with zero torsion and recurrent curvature without assuming that the recurrence covector W is not everywhere zero.

Beginning with §3 is the study of linear connexions with zero torsion and

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[1] Throughout this paper, the Riemannian connexions mentioned are those with a definite or indefinite metric. While Riemannian connexions with a positive definite metric and $\nabla R=0$ were described by E. Cartan as "une classe remarquable d'espaces de Riemann," the interesting case of Riemannian connexions with $\nabla R=W\otimes R$ and $W\not\equiv 0$ is when the metric is indefinite.

recurrent curvature for which the recurrence covector W is not everywhere zero. In §3.1 we give some examples of such linear connexions showing the nature of the subset $M_0 = \{u \in M: W(u) = 0\}$. In §§3.2–3.3 we prove among other things that the tensor ∇W is either everywhere symmetric or nowhere symmetric, and that the holonomy group can, and can only, be of dimension $1, 2, \cdots, n-1$. In §§4.1–4.3 we study the curvature tensor R in detail. Local and global decompositions of R in $M \backslash M_0$ (i.e. in the complement of M_0 in M) are obtained which give rise to a number of local fields of vectors and covectors in $M \backslash M_0$. Existence of certain global parallel fields of planes or coplanes spanned by these local vectors or covectors are established in §§5.1–5.3. There we also prove that the Ricci tensor is of (constant) rank $\leqq 2$, and that in the case where the tensor ∇W is symmetric, the condition for the holonomy group to be abelian is expressible in terms of the pseudo-orthogonality between certain parallel fields of planes and coplanes. The paper ends in §§6.1–6.3 with a study of a decomposition of the tensor ∇W and the fields of covectors or coplanes arising from it.

In the course of our investigation, it is found that the properties of the linear connexion are quite different in the two cases where the dimension of the holonomy group is greater than 1 or equal to 1. The latter case corresponds to the "simple" type (Walker [8]) of Riemannian connexion with recurrent curvature. It would be interesting to specialize the results presented in this paper to the case of Riemannian connexions and compare the results thus obtained with the earlier local results of H. S. Ruse, A. G. Walker and V. Hlavaty.

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1. Preliminaries

Parallel fields of planes and recurrent tensors on a C^∞-manifold with a linear connexion have been studied by A. G. Walker [7] and the author (Wong [10; 13]). Here, emphasizing the global aspects, we summarize a number of definitions and known results and add a few new ones for later use.

1.1. **Parallel fields of vectors along a curve and parallel fields of planes or coplanes on M.** Let M be a connected, n-dimensional C^∞-manifold (which consequently is also arcwise connected). A *plane* (resp. *coplane*) at a point $u \in M$ is a linear space spanned by a number of vectors (resp. covectors) at u. A *field of planes* (resp. *coplanes*) on M is a C^∞-assignment of a plane (resp. coplane) to each point of M.

Let $C: u = u(\tau)$, $0 \leqq \tau \leqq 1$, be a sectionally C^∞-curve in M. It is easy to see that C can be divided into a finite number of C^∞-sections each of which lies in a coordinate neighborhood. Let (U, u^h) be any one of these coordinate neighborhoods, and $u^h = u^h(\tau)$, $\tau_1 \leqq \tau \leqq \tau_2$, the C^∞-section of C lying in U. Then a vector field $X(\tau)$ defined on C is said to be *parallel* along C if the components $X^h(\tau)$ of $X(\tau)$ in each (U, u^h) satisfy the linear differential equations

$$\frac{du^l}{d\tau} \nabla_l X^h = 0, \quad \text{i.e.} \quad \frac{dX^h}{d\tau} + \frac{du^j}{d\tau} \Gamma_{ji}{}^h X^i = 0 \qquad (\tau_1 \leqq \tau \leqq \tau_2).$$

The following lemmas are easy consequences of well-known properties of the solutions of systems of linear differential equations:

LEMMA 1.1. *For any given vector A at the point $u(0)$, there exists a unique parallel vector field $X(\tau)$ along C such that $X(0) = A$.*

LEMMA 1.2. *If X_α $(1 \leqq \alpha \leqq n)$ are any n parallel vector fields along C and X_α are independent at some point of C, then X_α are everywhere independent along C. In this case, any parallel vector field X along C can be expressed uniquely as $X = c^\alpha X_\alpha$, where c^α are constants.*

LEMMA 1.3. *If X_A $(1 \leqq A \leqq r)$ are any r parallel vector fields along C, then the number of independent vectors among $X_A(\tau)$ at any point of C is constant along C.*

A field D of planes on M is *parallel* if, for any two points $u_0, u_1 \in M$, a vector in $D(u_0)$ is displaced parallelly along any (sectionally C^∞-) curve joining u_0 to u_1 into a vector in $D(u_1)$. Covector fields which are parallel along a curve and parallel fields of coplanes on M are defined in a similar way.

On account of Lemma 1.3, the planes of a parallel field D on M must be of the same dimension everywhere on M. Thus, if we wish to indicate the dimension of the planes, we can say that D is a parallel field of r-planes.

Using Lemma 1.3, we can easily prove

LEMMA 1.4. *Let D_1 be a parallel field of r_1-planes and D_2 a parallel field of r_2-planes on M. Then the intersection planes $D_1(u) \cap D_2(u)$ at all points $u \in M$ are of the same dimension, and they form a parallel field of planes on M. The same is also true of the union planes $D_1(u) \cup D_2(u)$.*

The following well-known result can be proved by modifying Walker's [7, Theorem 3.3] original proof in the case of Riemannian connexion:

LEMMA 1.5. *A field D of r-planes on M is parallel iff in each coordinate neighborhood $U \subset M$ and for any local basis X_A^h $(1 \leqq A, B \leqq r)$ of D in U,*

$$\nabla_l X_B^h = L_{lB}^A X_A^h,$$

where L_{lB}^A are some covectors on U.

We are now ready to prove our main result in this paragraph.

THEOREM 1.1. *Let r be a fixed positive number. If for each point u in M, there exists some coordinate neighborhood $U \ni u$ in M, and a set of r vectors Y^h_A $(1 \leqq A \leqq r)$ on U such that*

$$\text{(1.1)} \qquad \text{on } U\colon \nabla_l Y^h_A = L^B_{lA} Y^h_B \qquad (1 \leqq A, B \leqq r),$$

and

$$\text{(1.2)} \qquad \text{on } U \cap U^*\colon \overset{*}{Y}{}^h_{A^*} = \phi^A_{A^*} Y^h_A, \qquad Y^h_A = \phi^{A^*}_A \overset{*}{Y}{}^h_{A^*} \qquad (1 \leqq A, A^* \leqq r),$$

where U^* is another coordinate neighborhood and $\phi^A_{A^*}$, $\phi^{A^*}_A$ are functions on $U \cap U^*$, then the local fields of planes spanned by these sets of local vector fields piece together into a parallel field of planes on M.

Proof. Condition (1.2) assures us that the local fields of planes spanned by the sets of local vector fields piece together to form a field of planes on M. Therefore, on account of Lemma 1.5, it is sufficient to prove that if the r vectors Y^h_A on U satisfy condition (1.1), the number of independent vectors among them is the same everywhere in U.

To prove this, consider in U any two points u_0, u_1, and some curve $u(\tau)$, $0 \leqq \tau \leqq 1$, joining $u_0 = u(0)$ to $u_1 = u(1)$. Let X^α_h $(1 \leqq \alpha \leqq n)$ be some parallel coframe along $u(\tau)$, and define the functions $f^\alpha_A = Y^h_A X^\alpha_h$ on $u(\tau)$. We have, by condition (1.1),

$$\begin{aligned} d_\tau(Y^h_A X^\alpha_h) &= \delta_\tau(Y^h_A X^\alpha_h) = (\delta_\tau Y^h_A) X^\alpha_h \\ &= (d_\tau u^l) L^B_{lA} Y^h_B X^\alpha_h, \end{aligned}$$

where $d_\tau = d/d\tau$ and $\delta_\tau = (d_\tau u^l)\nabla_l$. Therefore, the functions f^α_A satisfy the equations

$$d_\tau f^\alpha_A = (d_\tau u^l) L^B_{lA} f^\alpha_B.$$

In other words, $f^\alpha_A(\tau)$, $1 \leqq \alpha \leqq n$, are n solutions of the system of linear differential equations

$$d_\tau f_A = (d_\tau u^l) L^B_{lA} f_B$$

in the r unknown functions f_A $(1 \leqq A \leqq r)$.

Let $\bar{f}^B_A$ $(1 \leqq B \leqq r)$ be any r independent solutions of this system of linear differential equations. Then $f^\alpha_A = \bar{f}^B_A c^\alpha_B$, where c^α_B are constants. Therefore, since the matrix $(\bar{f}^B_A)$ is of rank r all along $u(\tau)$ (from the theory of linear differential equations),

$$\operatorname{rank}\,(f^\alpha_A) = \operatorname{rank}\,(c^\alpha_B) = \text{constant along } u(\tau).$$

But $Y_A^h X_h^\alpha = f_A^\alpha$ and matrix (X_h^α) is of rank n all along $u(\tau)$. Therefore, rank $(Y_A^h) = \text{rank } (f_A^\alpha) = \text{constant}$ along $u(\tau)$. Hence

$$\text{rank } (Y_A^h)_{u_0} = \text{rank } (Y_A^h)_{u_1},$$

as was to be proved.

It is easy to see that results similar to Lemmas 1.1–1.5 and Theorem 1.1 hold for covector fields and fields of coplanes.

A vector X^h and a covector Y_h at a point $u \in M$ are said to be *pseudo-orthogonal* to each other if $X^h Y_h = 0$. Pseudo-orthogonality between an r-plane and an r'-coplane at a point is defined in an obvious manner. Moreover, it is easy to prove

LEMMA 1.6. *Let D be a parallel field of r-planes on M and D' a parallel field of r'-coplanes on M. Then if D and D' are pseudo-orthogonal at some point of M, they are pseudo-orthogonal at every point of M.*

LEMMA 1.7. *Let D be a field of r-planes on M and D' the field of $(n-r)$-coplanes pseudo-orthogonal to D. Then if D is parallel, D' is also parallel, and conversely.*

1.2. **Recurrent tensors and holonomy group of a linear connexion with recurrent curvature.** Let M be a connected n-dimensional C^∞-manifold on which a linear connexion has been given. Then we have

THEOREM 1.2. *If S is any tensor, say of type* (1, 3), *on M satisfying the condition $\nabla S = W \otimes S$, then*

(a) *To each point u of M, we can assign* (*not necessarily in a continuous manner*) *a frame $X_\alpha^h(u)$ such that the set of n^4 numbers*

$$S_{\beta\alpha\mu}{}^{\lambda}(u) = (S_{kji}{}^{h} X_\beta^k X_\alpha^j X_\mu^i X_h^\lambda)(u), \tag{1.3}$$

which are functions of u, are proportional to a set of n^4 constants.

(b) *S is either everywhere zero or nowhere zero.*

A tensor S on M is said to be *recurrent* if it satisfies the condition $\nabla S = W \otimes S$ without being zero everywhere. Then by Theorem 1.2 (b),

COROLLARY. *A recurrent tensor is nowhere zero.*

Proof of Theorem 1.2. The frame that we assign to each point of M is constructed as follows. Fix any point $u_0 \in M$ and assign to it any fixed frame $X_\alpha(u_0)$. For each point u_1 of M, fix a sectionally C^∞-curve $C_1: u = u(\tau)$, $-\epsilon \leqq \tau \leqq 1 + \epsilon$, passing through the points $u_0 = u(0)$, $u_1 = u(1)$. By displacing the frame $X_\alpha(u_0)$ at u_0 parallelly along C_1, we obtain a field of frames $Y_\alpha(\tau)$ along C_1. Then $Y_\alpha(1)$ is the frame $X_\alpha(u_1)$ that we assign to the point u_1. To prove (a) we have to prove that the two sets of numbers

$$S_{\beta\alpha\mu}{}^{\lambda}(u_1), \qquad S_{\beta\alpha\mu}{}^{\lambda}(u_0)$$

constructed by using the frames $X_\alpha(u_1)$ and $X_\alpha(u_0)$ in accordance with (1.3) are proportional. For this purpose, we now prove that the set of numbers

$$\tilde{S}_{\beta\alpha\mu}{}^{\lambda}(\tau) = (S_{kji}{}^{h}(u)\, \overset{k}{Y}_\beta \overset{j}{Y}_\alpha \overset{i}{Y}_\mu \overset{\lambda}{Y}_h)(\tau)$$

defined at each point of C_1 are always proportional to the set of numbers $\tilde{S}_{\beta\alpha\mu}{}^{\lambda}(0)$. Since the sectionally C^∞-curve C_1 can be divided into a finite number of C^∞-sections each lying in a coordinate neighborhood, it suffices to prove the last assertion for the case when the points u_0, u_1 lie in the same coordinate neighborhood and the curve C_1 through them is C^∞. Since $Y_\alpha(\tau)$ is parallel along C_1 and S satisfies the condition $\nabla S = W \otimes S$, we have

$$\begin{aligned} d_\tau \tilde{S}_{\beta\alpha\mu}{}^{\lambda} &= (d_\tau \overset{l}{u}) \nabla_l \tilde{S}_{\beta\alpha\mu}{}^{\lambda} \\ &= (d_\tau \overset{l}{u}) \nabla_l S_{kji}{}^{h}(u)\, \overset{k}{Y}_\beta \overset{j}{Y}_\alpha \overset{i}{Y}_\mu \overset{\lambda}{Y}_h \\ &= (d_\tau \overset{l}{u}) W_l(u) S_{kji}{}^{h}(u)\, \overset{k}{Y}_\beta \overset{j}{Y}_\alpha \overset{i}{Y}_\mu \overset{\lambda}{Y}_h \end{aligned}$$

i.e.,

$$d_\tau \tilde{S}_{\beta\alpha\mu}{}^{\lambda} = (d_\tau u^l) W_l(u) \tilde{S}_{\beta\alpha\mu}{}^{\lambda}.$$

From this it follows that

$$\tilde{S}_{\beta\alpha\mu}{}^{\lambda}(\tau) = \tilde{S}_{\beta\alpha\mu}{}^{\lambda}(0) \exp \int_0^\tau (d_\tau u^l) W_l(u) d\tau.$$

This shows that the set of n^4 numbers $S_{\beta\alpha\mu}{}^{\lambda}(u_1) = \tilde{S}_{\beta\alpha\mu}{}^{\lambda}(1)$ are proportional to the set of n^4 numbers $S_{\beta\alpha\mu}{}^{\lambda}(u_0) = \tilde{S}_{\beta\alpha\mu}{}^{\lambda}(0)$, which proves (a). In particular, if all the n^4 numbers $S_{\beta\alpha\mu}{}^{\lambda}(u_0)$ are zero, so are also all the n^4 numbers $S_{\beta\alpha\mu}{}^{\lambda}(u_1)$; in other words, if the tensor $S(u_0)$ is zero, so is the tensor $S(u_1)$. This proves (b).

Let u be a point in M and C a loop at u (i.e. a sectionally C^∞ closed curve beginning and ending at u). If we take any frame (u, X_α) at u and transport it parallelly along C once to arrive at the frame (u, X_α') at u, then the linear transformation σ_C in the tangent n-plane to M at u which carries the frame (u, X_α) into the frame (u, X_α') is independent of the choice of the initial frame (u, X_α). For a fixed point u in M, the set of linear transformations σ_C corresponding to all the possible loops at u form a subgroup of the general linear group $GL(n, R)$. This subgroup is the *holonomy group* of the linear connexion with reference point u. It follows at once from definition that the holonomy group at $u \in M$ leaves invariant an r-plane of the tangent n-plane to M at u iff there exists a parallel field of r-planes on M. Since the holonomy groups with different reference points are isomorphic, the reference point will not be mentioned specifically.

A linear connexion on M is said to be with *recurrent curvature* if its curvature tensor is recurrent. The following theorem is known (Wong [13, Theorem 4.3]):

THEOREM 1.3. *For a linear connexion with recurrent curvature on a connected n-dimensional C^∞-manifold M, the Lie algebra of its holonomy group H is spanned by the following elements (which are $n\times n$ matrices) of the Lie algebra of $GL(n, R)$:*

$$\textit{matrix } [R_{(\beta\alpha)\mu}{}^{\lambda}(u)], \qquad 1 \leqq \beta, \alpha \leqq n,$$

where

$$R_{(\beta\alpha)\mu}{}^{\lambda}(u) = (R_{kji}{}^{h} X^k_\beta X^j_\alpha X^i_\mu X^\lambda_h)(u),$$

u is any point in M and X^i_α any frame at u. Consequently, the dimension of the holonomy group H is equal to the number of independent ones among the matrices $[R_{(\beta\alpha)\mu}{}^{\lambda}(u)]$.

It follows from this that the holonomy group of a linear connexion with recurrent curvature on M is at most of dimension $n(n-1)/2$.

1.3. **Local decomposition of a recurrent tensor.** The main result in this paragraph is Theorem 1.4 concerning a local decomposition of a given recurrent tensor into the sum of tensor products of tensors. That theorem and Lemma 1.8 hold for a tensor of any type and for any two complementary sets of indices, but for simplicity and in anticipation of our applications, we state and prove them for a tensor $S_{kji}{}^h$ of type (1, 3) and for the complementary sets of indices k, j and i, h.

We first consider the case of tensors and vectors at a single point $u\in U\subset M$. Let $S_{kji}{}^h$ be any tensor of type (1, 3). In the linear space of tensors of type (0, 2), the linear subspace spanned by the n^2 tensors

$$S_{kj\mu}{}^{\lambda} = S_{kji}{}^{h} X^i_\mu X^\lambda_h \qquad (1 \leqq \mu, \lambda \leqq n),$$

where X^i_α is any frame and X^α_h its dual coframe, is obviously independent of the choice of the frame X^i_α. We call this linear subspace the *(k, j)-support* of S, and its dimension the *(k, j)-dimension* of S. In an analogous manner, the (i, h)-support, (i, h)-dimension or the j-support, etc. of S can be defined.

The following lemma is known and quite easy to prove (see, for example, Hlavaty [1, §1]).

LEMMA 1.8. *Let $S_{kji}{}^h$ be a tensor of type* (1, 3) *at the point $u\in U\subset M$. Then*

(a) *The (k, j)-dimension = the (i, h)-dimension.*

(b) *The (k, j)-dimension of S is equal to r iff S can be expressed as*

$$S_{kji}{}^{h} = P^{A}_{kj} Q_{Ai}{}^{h} \qquad (1 \leqq A \leqq r), \tag{1.4}$$

where P^A_{kj} (also $Q_{Ai}{}^h$) are r linearly independent tensors.

(c) *If the tensor S can also be expressed as*

$$S_{kji}{}^h = \overset{*}{P}{}^{A^*}_{kj}\overset{*}{Q}_{A^*i}{}^h, \tag{1.4*}$$

where $\overset{*}{P}{}^{A^*}_{kj}$ (also $\overset{*}{Q}_{A^*i}{}^h$) are r^* linearly independent tensors, then $r^*=r$ and

$$\overset{*}{P}{}^{A^*}_{kj} = \phi^{A^*}_A P^A_{kj}, \qquad \overset{*}{Q}_{A^*i}{}^h = \phi^A_{A^*}Q_{Ai}{}^h,$$

where $\phi^{A^*}_A$, $\phi^A_{A^*}$ are scalars satisfying $\phi^{A^*}_A\phi^B_{A^*}=\delta^B_A$.

(d) *If*
$$S_{\beta\alpha\mu}{}^\lambda = S_{kji}{}^h X^k_\beta X^j_\alpha X^i_\mu X^\lambda_h,$$

where X^h_α is any frame and X^α_h its dual coframe, then the (k, j)-dimension ($=$the (i, h)-dimension) of S is equal to the rank of the matrix

$$[S_{\beta\alpha\mu}{}^\lambda],$$

where (μ, λ) denote the row and (β, α) denote the column.

Proof. We omit the details of the proof, but merely indicate how the tensors P^A_{kj} and $Q_{Ai}{}^h$ can be constructed. Let r be the (k, j)-dimension of $S_{kji}{}^h$, and let P^A_{kj} $(1 \leqq A \leqq r)$ be a set of r independent tensors among the n^2 tensors $S_{kj\mu}{}^\lambda = S_{kji}{}^h X^i_\mu X^\lambda_h$ $(1 \leqq \mu, \lambda \leqq n)$. Then there exist scalars $\rho_{A\mu}{}^\lambda$ such that

$$S_{kj\mu}{}^\lambda = \rho_{A\mu}{}^\lambda P^A_{kj},$$

and

$$S_{kji}{}^h = (\rho_{A\mu}{}^\lambda P^A_{kj})X^\mu_i X^h_\lambda = P^A_{kj}(\rho_{A\mu}{}^\lambda X^\mu_i X^h_\lambda).$$

Thus, writing $\rho_{A\lambda}^\mu X^\mu_i X^h_\lambda$ as $Q_{Ai}{}^h$, we have a decomposition of the form (1.4).

We now consider the case of a recurrent tensor on M.

THEOREM 1.4. *Let S be a recurrent tensor of type (1, 3) on M: $S\neq 0$, $\nabla S = W\otimes S$, and let the components of S in any coordinate system (U, u^h) be $S_{kji}{}^h$. Then we have*

(a) *The (k, j)-dimension of S is constant on M; in other words, the (k, j)-dimension $r(u)$ of $S(u)$ defined at every point u of M is the same.*

(b) *In a suitable coordinate neighborhood U of each point u in M, S can be decomposed into the sum of tensor products*

$$S_{kji}{}^h = P^A_{kj}Q_{Ai}{}^h \qquad (1 \leqq A, B \leqq r), \tag{1.5}$$

where the r tensors P^A_{kj} (also the r tensors $Q_{Ai}{}^h$) are everywhere independent in U.

(c) *The tensors P^A_{kj} and $Q_{Ai}{}^h$ satisfy the relations:*

$$\nabla_l P_{kj}^{A} = L_{lB}^{A} P_{kj}^{B}, \qquad \nabla_l Q_{Bi}{}^{h} = \tilde{L}_{lB}^{A} Q_{Ai}{}^{h}, \tag{1.6}$$
$$L_{lB}^{A} + \tilde{L}_{lB}^{A} = W_l \delta_B^A,$$

where L_{lB}^A, $\tilde{L}_{lB}^A$ *are some covectors.*

(d) *If on another coordinate neighborhood* $U^* \ni u$, S *is decomposed into*

$$S_{kji}{}^{h} = \overset{*}{P}{}_{kj}^{A^*} \overset{*}{Q}{}_{A^*i}{}^{h} \qquad (1 \leqq A, B, A^* \leqq r),$$

then on $U \cap U^*$,

$$\overset{*}{P}{}_{kj}^{A^*} = P_{kj}^{A} \phi_A^{A^*}, \qquad \overset{*}{Q}{}_{A^*i}{}^{h} = \phi_{A^*}^{A} Q_{Ai}{}^{h},$$

where $\phi_A^{A^*}$, $\phi_{A^*}^A$ *are scalars on* $U \cap U^*$ *such that* $\phi_A^{A^*} \phi_{A^*}^B = \delta_A^B$.

Proof. By Lemma 1.8(d), the (k, j)-dimension $r(u)$ of the tensor $S(u)$ is equal to the rank of the matrix

$$[S_{\beta\alpha\mu}{}^{\lambda}(u)] \qquad \text{with row } (\mu\lambda) \text{ and column } (\beta\alpha),$$

where $S_{\beta\alpha\mu}{}^{\lambda}(u) = (S_{kji}{}^{h} X_\beta^k X_\alpha^j X_\mu^i X_h^\lambda)(u)$ and X_α^i is any frame at u. Since S is recurrent, we can, by Theorem 1.2(a) and its corollary, assign a frame to each point u of M so that the set of numbers $S_{\beta\alpha\mu}{}^{\lambda}(u)$, not all zero, are proportional to a set of constants, say $c_{\beta\alpha\mu}{}^{\lambda}$, which are independent of u. Therefore, $r(u)$ is equal to the rank of the matrix $[c_{\beta\alpha\mu}{}^{\lambda}]$ and is consequently constant on M. This proves (a).

To prove (b), we take any C^∞ field of frames X_α^i in any coordinate neighborhood $\tilde{U}$ containing u and use it to construct the tensors P_{kj}^A, $Q_{Ai}{}^h$ as in the proof of Lemma 1.8. These tensors are defined on $\tilde{U}$ and are obviously C^∞. They will satisfy the independence condition in some suitable neighborhood $U \subset \tilde{U}$ of u.

Finally, to prove (c), we differentiate (1.5) covariantly and get

$$(\nabla_l P_{kj}^{A}) Q_{Ai}{}^{h} + P_{kj}^{A} (\nabla_l Q_{Ai}{}^{h}) = \nabla_l S_{kji}{}^{h} = W_l S_{kji}{}^{h} = W_l P_{kj}^{A} Q_{Ai}{}^{h}. \tag{1.7}$$

Since the r tensors P_{kj}^A (also the r tensors $Q_{Ai}{}^h$) are independent, there exist some tensors $\overline{P}_B^{kj}$ and $\overline{Q}_h^{Bi}$ such that $P_{kj}^A \overline{P}_B^{kj} = \delta_B^A$ and $Q_{Ai}{}^h \overline{Q}_h^{Bi} = \delta_A^B$. Contractions of equation (1.7) by $\overline{Q}_h^{Bi}$ and $\overline{P}_B^{kj}$ respectively show that $\nabla_l P_{kj}^A$ and $\nabla_l Q_{Bi}{}^h$ are of the form

$$\nabla_l P_{kj}^{A} = L_{lB}^{A} P_{kj}^{B}, \qquad \nabla_l Q_{Bi}{}^{h} = \tilde{L}_{lB}^{A} Q_{Ai}{}^{h}.$$

Substituting these in (1.7), we arrive at

$$L_{lB}^{A} + \tilde{L}_{lB}^{A} = W_l \delta_B^A,$$

which completes the proof of the theorem.

Theorem 1.4 holds for a recurrent tensor of any type and for any two complementary sets of indices. Thus, let s be the h-dimension of S and $Q_A{}^h$ any s vectors which span the h-support of S. Then S can be locally decomposed in U as

$$S_{kji}{}^h = P_{kji}^{A} Q_A{}^h \qquad (1 \leqq A, B \leqq s),$$

and the s vectors $Q_A{}^h$ satisfy the relations

$$\nabla_l Q_B{}^h = \tilde{L}_{lB}^{A} Q_A{}^h.$$

Furthermore, if

$$S_{kji}{}^h = \overset{*}{P}{}_{kji}^{A^*} \overset{*}{Q}_{A^*}{}^h \qquad (1 \leqq A^*, B^* \leqq s)$$

is a local decomposition of S on U^*, then on $U \cap U^*$,

$$\overset{*}{Q}_{A^*}{}^h = \phi_{A^*}^{A} Q_A{}^h, \qquad Q_A{}^h = \phi_A^{A^*} \overset{*}{Q}_{A^*}{}^h.$$

Hence the local vectors $Q_A{}^h$ have the properties of the vectors $Y_A{}^h$ in Theorem 1.1, and we obtain

COROLLARY. *If S is a recurrent tensor on M, and h any of its contravariant (resp. covariant) index, then the h-dimension of S is constant on M and the h-support forms a parallel field of planes (resp. coplanes) on M.*

2. LINEAR CONNEXIONS WITH $\nabla R = W \otimes R$

In this section, we give a few easy but indispensable results concerning linear connexions with recurrent curvature. Here *we do not assume that the recurrence covector W is not everywhere zero.*

2.1. **The Ricci tensor.** The Ricci tensor is defined by $R_{ji} = R_{hji}{}^h$. We have

THEOREM 2.1. *For a linear connexion with recurrent curvature (not necessarily without torsion),*

(a) *The Ricci tensor is everywhere symmetric (resp. skew-symmetric) or nowhere symmetric (resp. skew-symmetric).*

(b) *The Ricci tensor is of the constant rank on M; in particular, it is everywhere zero or nowhere zero (cf. Theorem 5.2).*

Proof. From

$$\nabla_l R_{kji}{}^h = W_l R_{kji}{}^h$$

we deduce

$$\nabla_l R_{ji} = W_l R_{ji}, \qquad \nabla_l(R_{ji} - R_{ij}) = W_l(R_{ji} - R_{ij}),$$
$$\nabla_l(R_{ji} + R_{ij}) = W_l(R_{ji} + R_{ij}).$$

On account of Theorem 1.2(b), the last two equations imply (a) of the theorem. Next, since the rank of R_{ji} is equal to the i-dimension of R_{ji}, (b) follows from the corollary to Theorem 1.4.

In this connection, it is useful to note that for *any linear connexion with zero torsion*,

$$R_{kjh}{}^{h} = -(R_{kj} - R_{jk}). \tag{2.1}$$

This is obtained by contracting the indices h and i in the first Bianchi identity (0.1).

2.2. **Supports of indices of the curvature tensor** (cf. §1.3).

THEOREM 2.2. *For the curvature tensor $R_{kji}{}^{h}$ of any linear connexion with zero torsion,*

$$j\text{-support} = k\text{-support}, \qquad i\text{-support} \subset k\text{-support}.$$

If the linear connexion is with recurrent curvature, the support of each of the indices of the curvature tensor $R_{kji}{}^{h}$ forms a parallel field of planes or coplanes on M.

Proof. The first assertion is a consequence of the identities $R_{kji}{}^{h} = -R_{jki}{}^{h}$ and (0.1) which give

$$R_{\beta j\mu}{}^{\lambda} = -R_{j\beta\mu}{}^{\lambda}, \qquad R_{\beta\alpha i}{}^{\lambda} = -R_{\alpha i\beta}{}^{\lambda} + R_{i\beta\alpha}{}^{\lambda},$$

where for example, $R_{\beta j\mu}{}^{\lambda} = R_{kji}{}^{h}X^{k}_{\beta}X^{i}_{\mu}X^{\lambda}_{h}$. The second assertion is a particular case of the Corollary to Theorem 1.4.

2.3. **The holonomy group.**

THEOREM 2.3. *For a linear connexion with recurrent curvature (not necessarily with zero torsion), the (k, j)-dimension of its curvature tensor R is constant on M. If the (k, j)-dimension of R is r, and R is decomposed at any point $u \in M$ into*

$$R_{kji}{}^{h} = P^{A}_{kj}Q_{Ai}{}^{h} \qquad (1 \leqq A \leqq r), \tag{2.2}$$

then the Lie algebra of the holonomy group H is spanned by the r independent elements: matrix $[Q_{A\mu}{}^{\lambda}]$, where $Q_{A\mu}{}^{\lambda} = Q_{Ai}{}^{h}X^{i}_{\mu}X^{\lambda}_{h}$ and X^{i}_{α} is any frame at u. Consequently, dim $H = r$.

Proof. The first assertion is a particular case of Theorem 1.4. To prove the second assertion, we write

$$R_{\beta\alpha\mu}{}^{\lambda} = R_{kji}{}^{h}X^{k}_{\beta}X^{j}_{\alpha}X^{i}_{\mu}X^{\lambda}_{h} = (P^{A}_{kj}X^{k}_{\beta}X^{j}_{\alpha})(Q_{Ai}{}^{h}X^{i}_{\mu}X^{\lambda}_{h})$$

as

$$R_{\beta\alpha\mu}{}^{\lambda} = P^{A}_{\beta\alpha}Q_{A\mu}{}^{\lambda}.$$

By Theorem 1.3, the Lie algebra of the holonomy group is spanned by the $n(n-1)/2$ elements

$$\text{matrix } [R_{(\beta\alpha)\mu}{}^{\lambda}] = P^{A}_{\beta\alpha} \text{ matrix } [Q_{A\mu}{}^{\lambda}].$$

Since the r tensors $Q_{Ai}{}^{h}$ (also the r tensors P^{A}_{kj}) at u are independent (cf. Theorem 1.4), it is easily seen from the above equation that the Lie algebra of H is spanned by the r independent elements: matrix $[Q_{A\mu}{}^{\lambda}]$, as was to be proved.

2.4. **A local decomposition of the curvature tensor.**

THEOREM 2.4. *For a linear connexion with zero torsion and recurrent curvature*: $T=0$, $\nabla R = W\otimes R$, *if r is the dimension of the holonomy group, then in a suitable coordinate neighborhood U of each point, the curvature tensor can be decomposed as*

$$R_{kji}{}^{h} = P^{A}_{kj}Q_{Ai}{}^{h} \qquad (1 \leqq A, B, C, D \leqq r), \tag{2.2}$$

where the r tensors P^{A}_{kj} (also the r tensors $Q_{Ai}{}^{h}$) are everywhere independent in U and satisfy the following relations:

$$\nabla_l P^{A}_{kj} = L^{A}_{lB}P^{B}_{kj}, \qquad \nabla_l Q_{Bi}{}^{h} = \tilde{L}^{A}_{lB}Q_{Ai}{}^{h}, \qquad L^{A}_{lB} + \tilde{L}^{A}_{lB} = W_l\delta^{A}_{B}; \tag{2.3}$$

$$\begin{aligned} &P^{A}_{a[k}Q_{Bj]}{}^{a} = C^{A}_{BC}P^{C}_{kj}, && \nabla_{[m}L^{A}_{l]B} + L^{D}_{[mB}L^{A}_{l]D} = C^{A}_{CB}P^{C}_{ml},\\ &Q_{[Ba}{}^{h}Q_{C]i}{}^{a} = \tilde{C}^{A}_{BC}Q_{Ai}{}^{h}, && \nabla_{[m}\tilde{L}^{A}_{l]B} + \tilde{L}^{A}_{[mD}\tilde{L}^{D}_{l]B} = \tilde{C}^{A}_{CB}P^{C}_{ml};\\ &(\nabla_{[m}W_{l]})\delta^{A}_{C} = (C^{A}_{BC} + \tilde{C}^{A}_{BC})P^{C}_{ml}, && \nabla_l\tilde{C}^{A}_{BC} = \tilde{L}^{D}_{l[B}\tilde{C}^{A}_{DC]} - \tilde{L}^{A}_{lD}\tilde{C}^{D}_{BC}. \end{aligned} \tag{2.4}$$

Here W_l is the recurrence covector, L^{A}_{lB} and $\tilde{L}^{A}_{lB}$ are covectors, C^{A}_{BC} and $\tilde{C}^{A}_{BC}(=-\tilde{C}^{A}_{CB})$ are scalars, and square brackets used with indices indicate that alternation is taken over the two outermost indices; *thus*

$$P^{A}_{a[k}Q_{Bj]}{}^{a} \equiv P^{A}_{ak}Q_{Bj}{}^{a} - P^{A}_{aj} - Q_{Bk}{}^{a}.$$

Proof. The relations (2.3) have been proved in Theorem 1.4. We deduce from $(2.3)_1$ that

$$\nabla_{[m}\nabla_{l]}P^{A}_{kj} = (\nabla_{[m}L^{A}_{l]B} + L^{A}_{[lC}L^{C}_{m]B})P^{B}_{kj}.$$

Applying to the left side the Ricci identity for a linear connexion with zero torsion, we have on account of (2.2)

$$\begin{aligned} \nabla_{[m}\nabla_{l]}P^{A}_{kj} &= -R_{mlk}{}^{a}P^{A}_{aj} - R_{mlj}{}^{a}P^{A}_{ka}\\ &= P^{B}_{ml}(-Q_{Bk}{}^{a}P^{A}_{aj} + Q_{Bj}{}^{a}P^{A}_{ak}). \end{aligned}$$

Since P^{A}_{kj} are independent tensors, comparison of the above two expressions for $\nabla_{[m}\nabla_{l]}P^{A}_{kj}$ gives $(2.4)_{1,2}$. Similarly, equations $(2.4)_{3,4}$ can be proved by using (2.2) and $(2.3)_2$. Equation $(2.4)_5$ is an immediate consequence of $(2.3)_3$ and $(2.4)_{2,4}$. The last equation $(2.4)_6$ follows from $(2.3)_2$ and $(2.4)_3$.

We now prove

COROLLARY. *The holonomy group is abelian iff the functions $\tilde{C}^A_{BC}$ appearing in equation $(2.4)_3$ are all zero at some point of M (then they are all zero everywhere in M).*

Proof. Let X^i_α be any frame at u and let $Q_{A\mu}{}^\lambda = Q_{Ai}{}^h(u)X^i_\mu X^\lambda_h$. Then equation $(2.4)_3$ at u is equivalent to

$$Q_{[B\nu}{}^\lambda Q_{C]\mu}{}^\nu = \tilde{C}^A_{BC}(u)Q_{A\mu}{}^\lambda.$$

In consequence of Theorem 2.3, these are the structure equations of the holonomy group H (with reference point u) so that $\tilde{C}^A_{BC}(u)$ are the structural constants. Thus H is abelian iff $\tilde{C}^A_{BC}(u)=0$. Equation $(2.4)_6$ confirms the fact that if $\tilde{C}^A_{BC}$ are all zero at some point u of M, they are all zero everywhere in M.

3. LINEAR CONNEXIONS WITH $T=0$, $\nabla R=W\otimes R$, $W\not\equiv 0$

From now on we shall study exclusively linear connexions with zero torsion and recurrent curvature *for which the recurrence covector W is not everywhere zero.* As before, the n-dimensional C^∞-manifold M on which such a linear connexion γ is defined is assumed to be connected, and so arcwise connected.

Let $M_0=\{u\in M: W(u)=0\}$. If M_0 is an n-dimensional subset of M, as is the case in Example 1 in §3.1, then the restriction of γ to M_0 is a linear connexion γ_0 with $T=0$ and $\nabla R=0$. Such a linear connexion γ_0 generalizes the symmetric (Riemannian) space of E. Cartan and has been studied quite extensively. But the subset M_0 may also be of dimension $n-1$ or $n-2$, as is the case in Examples 2 and 3 of §3.1. Apart from this, we know nothing about the nature of the subset $M_0\subset M$.

Let M_0 be as defined above, and let M_1 be any arcwise connected component of $M\backslash M_0$. Then M_1 is an n-dimensional connected C^∞-manifold and the restriction of γ to M_1 is a linear connexion γ_1 with $T=0$, $\nabla R=W\otimes R$ for which the recurrence covector W is nowhere zero. Since the results in the previous sections were derived for a connected C^∞-manifold we can apply them directly to M and to each of the connected components of $M\backslash M_0$ but not to $M\backslash M_0$ which may not be connected. In what follows, we shall obtain various properties of the linear connexion γ and properties of M implied by the existence of γ on it. Some of the results hold on the whole of M; others only on $M\backslash M_0$ or on any of its connected components; still a few others only on $(M\backslash M_0)\backslash M_{00}$, where $M_{00}=\{u\in(M\backslash M_0): [(\nabla_l W_{[k})W_{j]}](u)=0\}$.

3.1. Some examples.

EXAMPLE 1. On the Euclidean n-space M: $(-\infty<u^1,\cdots,u^n<+\infty)$, let

$\gamma[\theta]$ denote the linear connexion (with zero torsion) defined by the components

$$\Gamma_{11}^{\xi} = b_{\zeta}^{\xi} u^{\zeta}\theta, \qquad \Gamma_{\eta 1}^{\ 2} = \Gamma_{1\eta}^{\ 2} = a_{\eta\zeta}u^{\zeta}\theta,$$

$$\Gamma_{11}^{\ 2} = \frac{1}{2} a_{\eta\zeta}u^{\eta}u^{\zeta}\frac{d\theta}{du^1} + c_{\zeta}u^{\zeta}\theta, \qquad (3 \leqq \xi, \eta, \zeta \leqq n),$$

all the other $\Gamma_{ji}^{\ h}$ are zero,

where the function $\theta=\theta(u^1)$ is nowhere zero, and $a_{\eta\zeta}$, b^{ξ}, c_{ζ} are constants not all zero such that $a_{\eta\zeta}=a_{\zeta\eta}$ and $a_{\xi\eta}b_{\zeta}^{\xi}=a_{\xi\zeta}b_{\eta}^{\xi}$.

By straightforward computation or from Wong [9], it can easily be verified that $\gamma[\theta]$ has recurrent curvature and the recurrence covector is $W_l=\partial(\log\theta)/\partial u^l$.

Taking

$$\theta = \begin{cases} 1 & \text{for } u^1 \leqq 0, \\ 1+\exp(-1/u^1) & \text{for } u^1 > 0, \end{cases}$$

we obtain on M a linear connexion with zero torsion and recurrent curvature for which $M_0=\{u\in M: W(u)=0\}$ is the subset $u^1\leqq 0$, and $M\backslash M_0$ is the connected subset $u^1>0$.

Example 2. Let M and $\gamma[\theta]$ be as in Example 1. Taking $\theta=1+(u^1)^2$, we obtain on M a linear connexion with zero torsion and recurrent curvature for which the subset M_0 is the $(n-1)$-plane $u^1=0$ and $M\backslash M_0$ consists of the two connected components $u^1<0$ and $u^1>0$.

Example 3. On the Euclidean n-space M: $(-\infty<u^1, \cdots, u^n<+\infty)$, let $\gamma[\phi]$ denote the linear connexion (with zero torsion) defined by the components

$$\Gamma_{11}^{\xi} = b^{\xi}\partial\phi/\partial z, \qquad \Gamma_{\eta 1}^{2} = \Gamma_{1\eta}^{2} = a\, e_{\eta}\partial\phi/\partial z,$$

$$\Gamma_{11}^{2} = a\partial\phi/\partial u^1 + c\partial\phi/\partial z, \qquad (3 \leqq \xi, \eta, \zeta \leqq n),$$

all the other $\Gamma_{ji}^{\ h}$ are zero,

where a, b^{ξ}, c are constants not all zero; e_{ζ} are constants not all zero; $z=e_{\zeta}u^{\zeta}$, $\phi=\phi(u^1, z)$ is such that $\partial^2\phi/\partial z^2$ is nowhere zero. By straightforward computation or from Wong [9], it can easily be verified that $\gamma[\phi]$ has recurrent curvature and the recurrence covector is $W_l=\partial(\log\partial^2\phi/\partial z^2)/\partial u^l$. If we take as $\phi(u^1, z)$ any solution of the differential equation

$$\partial^2\phi/\partial z^2 = 1 + (u^1)^2 + (z)^2,$$

we obtain on M a linear connexion with zero torsion and recurrent curvature for which the subset M_0 is the $(n-2)$-plane: $u^1=0=z$, and $M\backslash M_0$ is a connected subset.

3.2. **The recurrence covector W.** We now prove

THEOREM 3.1. *For a linear connexion with $T=0$, $\nabla R=W\otimes R$, $W\neq 0$, the recurrence covector W satisfies the equations*

$$\nabla_q(\nabla_{[m}W_{l]}) = W_q(\nabla_{[m}W_{l]}), \tag{3.1}$$

$$W_q\nabla_{[m}W_{l]} + W_m\nabla_{[l}W_{q]} + W_l\nabla_{[q}W_{m]} = 0. \tag{3.2}$$

Equation (3.1) implies that the tensor ∇W on M is either everywhere symmetric or nowhere symmetric; in the former case, W is locally a gradient. Equation (3.2) means that W is locally proportional to a gradient; in other words, the total differential equation $W_k du^k=0$ which is globally defined on M is locally completely integrable.

Proof. On account of $\nabla R=W\otimes R$, we have

$$\nabla_m\nabla_l R_{kji}{}^h = \nabla_m(W_l R_{kji}{}^h) = (\nabla_m W_l)R_{kji}{}^h + W_m W_l R_{kji}{}^h.$$

Therefore, by the Ricci identity,

$$\begin{aligned}(\nabla_{[m}W_{l]})R_{kji}{}^h &= \nabla_{[m}\nabla_{l]}R_{kji}{}^h \\ &= R_{mlp}{}^h R_{kji}{}^p - R_{mli}{}^p R_{kjp}{}^h - R_{mlj}{}^p R_{kpi}{}^h - R_{mlk}{}^p R_{pji}{}^h.\end{aligned}$$

(It is easy to verify that this equation is equivalent to $(2.4)_5$.) Applying ∇_q to both sides of this equation and using the latter again in the result, we get

$$\nabla_q(\nabla_{[m}W_{l]})R_{kji}{}^h + W_q(\nabla_{[m}W_{l]})R_{kji}{}^h = 2W_q(\nabla_{[m}W_{l]})R_{kji}{}^h,$$

i.e.,

$$\nabla_q(\nabla_{[m}W_{l]})R_{kji}{}^h = W_q(\nabla_{[m}W_{l]})R_{kji}{}^h.$$

Since the curvature tensor is nowhere zero, it follows that

$$\nabla_q(\nabla_{[m}W_{l]}) = W_q(\nabla_{[m}W_{l]}),$$

which proves (3.1). By Theorem 1.2(b), equation (3.1) implies that $\nabla_{[m}W_{l]}$ is either everywhere zero or nowhere zero.

To prove (3.2), let us denote by cyc( ) the cyclic expression with typical term inside the brackets. Then on account of (3.1), we have

$$\begin{aligned}\mathrm{cyc}(W_q\nabla_{[m}W_{l]}) &= \mathrm{cyc}(\nabla_q\nabla_{[m}W_{l]}) = \mathrm{cyc}(\nabla_{[m}\nabla_{l]}W_q) \\ &= \mathrm{cyc}(R_{mlq}{}^h W_h) \\ &= (R_{mlq}{}^h + R_{lqm}{}^h + R_{qml}{}^h)W_h \\ &= 0.\end{aligned}$$

This is (3.2) which is the well-known condition for W to be locally proportional to a gradient. Hence our theorem is completely proved.

In consequence of Theorems 2.1 and 3.1, the Ricci tensor and the tensor ∇W share the property that it is either everywhere symmetric or nowhere symmetric. We observe also that

(a) For a Riemannian connexion with $\nabla R = W\otimes R$ and $W\neq 0$, ∇W is symmetric (Walker [8, §6]).
(b) For a subflat or projectively flat connexion with $T=0$, $\nabla R = W\otimes R$ and $W\neq 0$, ∇W is symmetric iff the Ricci tensor is symmetric (Wong [11] and Wong and Yano [12]).
(c) For a 2-dimensional linear connexion with $T=0$, $\nabla R = W\otimes R$ and $W\neq 0$, ∇W is symmetric iff the Ricci tensor is symmetric (Wong [14]).

These facts support the truth of the following conjecture though the author has not been able to prove it.

CONJECTURE. *For every linear connexion with $T=0$, $\nabla R = W\otimes R$ and $W\neq 0$, the tensor ∇W is (everywhere) symmetric iff the Ricci tensor is (everywhere) symmetric.*

3.3. **Dimension of the holonomy group.**

LEMMA 3.1. *In any coordinate neighborhood U in which W has no zero, the curvature tensor can be expressed as*

$$R_{kji}{}^h = W_k S_{ji}{}^h - W_j S_{ki}{}^h,$$

where $S_{ji}{}^h$ is some tensor on U.

Proof. Using $\nabla R = W\otimes R$ in the second Bianchi identity (0.2), we get

$$W_l R_{kji}{}^h + W_k R_{jli}{}^h + W_j R_{lki}{}^h = 0.$$

Since W has no zero in U, there exists some vector X^l on U such that $X^l W_l = 1$. Contraction of the above equation by X^l gives

$$R_{kji}{}^h = W_k R_{lji}{}^h X^l - W_j R_{lki}{}^h X^l,$$

i.e.,

$$R_{kji}{}^h = W_k S_{ji}{}^h - W_j S_{ki}{}^h,$$

where we have put

$$S_{ji}{}^h = R_{lji}{}^h X^l.$$

We now prove

THEOREM 3.2. *Let H denote the holonomy group of a linear connexion with $T=0$, $\nabla R = W\otimes R$ and $W\neq 0$. Then $1\leqq \dim H\leqq n-1$. Moreover, for each integer r satisfying $1\leqq r\leqq n-1$, there exist linear connexions of this type for which* $\dim H=r$.

Proof. Let us consider a point $u\in M$ at which $W\neq 0$. By Lemma 3.1, the curvature tensor is of the form

$$R_{kji}{}^h = W_k S_{ji}{}^h - W_j S_{ki}{}^h$$

at u. Choose a frame X^h_α at u such that $W_\beta = W_k X^k_\beta = \delta^1_\beta$. Then the numbers $R_{(\beta\alpha)\mu}{}^\lambda(u)$ which appear in Theorem 1.3 are

$$R_{(\beta\alpha)\mu}{}^{\lambda}(u) = \overset{1}{\delta_\beta} S_{\alpha\mu}{}^{\lambda}(u) - \overset{1}{\delta_\alpha} S_{\beta\mu}{}^{\lambda}(u).$$

Consequently, among the $n(n-1)/2$ matrices

$$[R_{(\beta\alpha)\mu}{}^{\lambda}(u)] \qquad (1 \leqq \beta < \alpha \leqq n)$$

only the following $n-1$ can possibly be independent:

$$[R_{(1\alpha)\mu}{}^{\lambda}(u)] = [S_{(\alpha)\mu}{}^{\lambda}(u)] \qquad (2 \leqq \alpha \leqq n).$$

Therefore, by Theorem 1.3, dim $H \leqq n-1$. On the other hand, since the numbers $R_{(\beta\alpha)\mu}{}^{\lambda}(u)$ cannot all be zero, at least one of the above $n-1$ matrices is nonzero. Therefore dim $H \geqq 1$.

To prove the second half of the theorem, we consider some examples. On the Euclidean n-space M: $(-\infty < u^1, \cdots, u^n < +\infty)$, let γ be the linear connexion with zero torsion and recurrent curvature defined by the components

$$\Gamma_{11}^{3} = \theta u^{3}, \cdots, \qquad \Gamma_{11}^{r+2} = \theta u^{r+2},$$

all the other $\Gamma_{ji}{}^{h}$ are zero,

where $\theta = 1 + (u^1)^2$, and r is any fixed integer such that $1 \leqq r \leqq n-2$. This is a special case of Example 2 in §3.1. An easy computation will show that the only nonzero components of the curvature tensor are

$$R_{1j1}{}^{h} = \theta \delta_j^h \qquad (3 \leqq h, j \leqq r+2)$$

and those differing from these by sign. Taking $X_\alpha^h = \delta_\alpha^h$, we see by Theorem 1.3 that dim H is equal to the number of independent matrices among the following r $n \times n$ matrices:

$$[R_{(1\alpha)\mu}{}^{\lambda}] = \theta[\overset{1}{\delta_\mu} \overset{\lambda}{\delta_\alpha}] \qquad (\alpha = 3, \cdots, r+2).$$

Now in the αth matrix the $(\alpha, 1)$-element is its only nonzero element. Therefore, these r matrices are independent, and so dim $H = r$. Hence there exist linear connexions of the type under considerations for which dim $H = 1, 2, \cdots, n-2$.

To give an example for which dim $H = n-1$, let us consider on a suitable neighborhood U in M: $(-\infty < u^1, \cdots, u^n < +\infty)$ a projectively flat connexion with recurrent curvature. In Wong [12], we have shown that for such a linear connexion, the curvature tensor is of the form (in suitable coordinates)

$$R_{kji}{}^{h} = \theta(\delta_k^h \xi_j - \delta_j^h \xi_k)\xi_i \tag{3.3}$$

or

$$(3.4)\qquad R_{kji}{}^{h} = \psi[(\delta_k^h \xi_j - \delta_j^h \xi_k)\eta_i - (\xi_k \eta_j - \xi_j \eta_k)\delta_i^h] \qquad (\xi_i, \eta_i \text{ not proportional}),$$

according as its Ricci tensor is symmetric or nonsymmetric.

For the case (3.3), let us choose a frame X_α^h at $u \in U$ such that $\xi_\alpha = \xi_h X_\alpha^h = \delta_\alpha^1$. Then by Theorem 1.3, dim H is equal to the number of independent matrices among the following $n-1$ $n \times n$ matrices

$$[R_{(\beta 1)\mu}{}^{\lambda}(u)] = \theta[\delta_\beta^\lambda \xi_\mu(u)] \qquad (\beta = 2, \cdots, n).$$

But these $n-1$ matrices are all linearly independent since

$$\overset{\beta}{c}(\delta_\beta^\lambda \xi_\mu) = 0 \Rightarrow \overset{\beta}{c}\xi_\mu = 0 \Rightarrow \overset{\beta}{c} = 0.$$

Therefore, dim $H = n-1$.

For the case (3.4), ξ_i and η_i are not proportional. Then using a frame X_α^h at $u \in U$ such that

$$\xi_\alpha = \xi_h X_\alpha^h = \overset{1}{\delta}_\alpha, \qquad \eta_\alpha = \eta_h X_\alpha^h = \overset{2}{\delta}_\alpha,$$

we can verify that in this case also dim H is equal to $n-1$.

Hence we have not only completed the proof of Theorem 3.5 but also proved the following

THEOREM 3.3. *The holonomy group of an n-dimensional projectively flat connexion with $T=0$, $\nabla R = W \otimes R$ and $W \neq 0$ is of dimension $n-1$.*

In this connection, it is interesting to note that combining our Theorem 1.3 with a local result of Hlavaty [2, Theorem 3.4 a, b] or with certain equivalent results of Walker ([8, (7.2) and p. 55]), we have

THEOREM 3.4. *The holonomy group of an n-dimensional ($n>2$) Riemannian connexion with $\nabla R = W \otimes R$ with $W \neq 0$ is of dimension $\leqq n-2$.*

4. THE CURVATURE TENSOR

In this section, certain local and global decompositions of the curvature tensor R into sums of tensor products are proved and studied in detail. In the course of our investigation, the distinction between the cases where the holonomy group is of dimension >1 or $=1$ arises naturally. In addition to being of interest on their own, the results obtained will form a basis of our work in the next two sections.

4.1. A local decomposition of R in $M \backslash M_0$.

THEOREM 4.1. *Let a linear connexion on M be such that $T=0$, $\nabla R = W \otimes R$ and $W \neq 0$ and let $M_0 = \{u \in M \colon W(u) = 0\}$. Then on each coordinate neighborhood U in $M \backslash M_0$ there exists some tensor $S_{ji}{}^{h}$, symmetric in j, i and having no zero in U such that R is decomposed into*

$$R_{kji}{}^{h} = W_{k}S_{ji}{}^{h} - W_{j}S_{ki}{}^{h}. \tag{4.1}$$

The most general tensor $\tilde{S}_{ji}{}^{h}$ *having the properties of the tensor* $S_{ji}{}^{h}$ *is of the form*

$$\tilde{S}_{ji}{}^{h} = S_{ji}{}^{h} + W_{j}W_{i}C^{h}, \tag{4.2}$$

where C^{h} *is an arbitrary vector on* U.

Proof. Since W is nowhere zero in $M\backslash M_0$ it follows from Lemma 3.1 that on each coordinate neighborhood U of $M\backslash M_0$ there exists some tensor S of type (1, 2) such that

$$R_{kji}{}^{h} = W_{k}S_{ji}{}^{h} - W_{j}S_{ki}{}^{h} \tag{4.1}$$

holds.

If $\tilde{S}$ is any tensor of type (1, 2) on U satisfying

$$R_{kji}{}^{h} = W_{k}\tilde{S}_{ji}{}^{h} - W_{j}\tilde{S}_{ki}{}^{h}, \tag{4.$\tilde{1}$}$$

then we derive from this and (4.1) that

$$W_{k}(\tilde{S}_{ji}{}^{h} - S_{ji}{}^{h}) = W_{j}(\tilde{S}_{ki}{}^{h} - S_{ki}{}^{h}),$$

which gives (since W has no zero in U)

$$\tilde{S}_{ji}{}^{h} = S_{ji}{}^{h} + W_{j}B_{i}{}^{h}, \tag{4.3}$$

where $B_{i}{}^{h}$ is some tensor of type (1, 1) on U.

Conversely, if $S_{ji}{}^{h}$ is any tensor satisfying (4.1) and B_{i}^{h} is any tensor of type (1, 1), then the tensor $\tilde{S}_{ji}{}^{h}$ defined by (4.3) satisfies (4.1).

Now using (4.1) in the first Bianchi identity:

$$R_{kji}{}^{h} + R_{jik}{}^{h} + R_{ikj}{}^{h} = 0,$$

we obtain

$$(W_{k}S_{ji}{}^{h} - W_{j}S_{ki}{}^{h}) + (W_{j}S_{ik}{}^{h} - W_{i}S_{jk}{}^{h}) + (W_{i}S_{kj}{}^{h} - W_{k}S_{ij}{}^{h}) = 0,$$

i.e.

$$W_{k}S_{[ji]}{}^{h} + W_{j}S_{[ik]}{}^{h} + W_{i}S_{[kj]}{}^{h} = 0.$$

From this it follows easily that

$$S_{ji}{}^{h} - S_{ij}{}^{h} = - W_{j}B_{i}{}^{h} + W_{i}B_{j}{}^{h},$$

i.e.

$$S_{ji}{}^{h} + W_{j}B_{i}{}^{h} = S_{ij}{}^{h} + W_{i}B_{j}{}^{h}, \tag{4.4}$$

where $B_{i}{}^{h}$ is some tensor of type (1, 1) on U. Let us put $\tilde{S}_{ji}{}^{h} = S_{ji}{}^{h} + W_{j}B_{i}{}^{h}$. Then on account of (4.4), the tensor $\tilde{S}_{ji}{}^{h}$ is symmetric in j, i. Moreover, by the assertion below (4.3), it satisfies equation (4.$\tilde{1}$). Thus, we have shown that there exists a tensor having the properties of the tensor $S_{ji}{}^{h}$ in the theorem.

To complete the proof of the theorem, we see from (4.3) that the most general of such tensors is of the form

$$\tilde{S}_{ji}{}^{h} = S_{ji}{}^{h} + W_j B_i{}^{h}.$$

But, because $\tilde{S}_{ji}{}^{h} - S_{ji}{}^{h}$ is symmetric in j, i, the tensor $B_i{}^{h}$ must satisfy the equation $W_j B_i{}^{h} = W_i B_j{}^{h}$, and is therefore of the form $B_i{}^{h} = W_i C^h$, where C^h is an arbitrary vector. Hence, the most general tensor $\tilde{S}_{ji}{}^{h}$ having the properties of the tensor $S_{ji}{}^{h}$ in the theorem is

$$\tilde{S}_{ji}{}^{h} = S_{ji}{}^{h} + W_j W_i C^h,$$

as was to be proved.

4.2. **A global decomposition of the curvature tensor on $M \backslash M_0$.** The results in Theorem 4.1 enable us to prove the following

THEOREM 4.2. *Let a linear connexion on M be such that $T=0$, $\nabla R = W \otimes R$ and $W \not\equiv 0$, and let M_0 be the subset $\{u \in M: W(u) = 0\}$. Then there exists on $M \backslash M_0$ some tensor S of type* (1, 2) *which is*

(a) *symmetric in the two covariant indices,*

(b) *nowhere zero in $M \backslash M_0$, and*

(c) *such that on every coordinate neighborhood U in $M \backslash M_0$, R can be decomposed into*

$$R_{kji}{}^{h} = W_k S_{ji}{}^{h} - W_j S_{ki}{}^{h}.$$

Any two such tensors S on $M \backslash M_0$ differ from each other by a tensor of the form $W \otimes W \otimes C$, where C is some vector on $M \backslash M_0$.

Proof. Consider the bundle B^S of all tensors S of type (1, 2) at all points of $M \backslash M_0$ having the properties of the tensor $S_{ji}{}^{h}$ in Theorem 4.1. Because the most general S of such tensors at a point $u \in M \backslash M_0$ is of the form (4.2), the fiber F_u of B^S over u is a linear space isomorphic to R^n, and is therefore solid. Then, by a theorem of Steenrod's [6, p. 55] on existence of cross sections on fiber bundles, C^∞ cross sections of B^S exist. Any such cross section is a tensor S of type (1, 2) on $M \backslash M_0$ satisfying the conditions stated in the theorem. The last assertion in the theorem follows from (4.2), and this completes the proof.

4.3. **Another local decomposition of the curvature tensor in $M \backslash M_0$.**

THEOREM 4.3. *Let a linear connexion on M be such that $T=0$, $\nabla R = W \otimes R$ and $W \not\equiv 0$, and let $M_0 = \{u \in M: W(u) = 0\}$. If the dimension of the holonomy group is r $(1 \leqq r \leqq n-1)$, then for each point u of $M \backslash M_0$, there exists some coordinate neighborhood $U \ni u$ on which the curvature tensor can be decomposed into*

$$R_{kji}{}^{h} = (W_k W_j^{A} - W_j W_k^{A})(W_i Z_A^{h} + W_i^{B} Z_{AB}^{h}) \qquad (2 \leqq A, B \leqq r+1), \tag{4.5}$$

where W^A_k, Z^h_A, $Z^h_{AB}=Z^h_{BA}$ are covectors and vectors on U such that the $r+1$ covectors W_k, W^A_k are everywhere linearly independent.

Proof. Comparing the two expressions

$$(2.2) \qquad R_{kji}{}^h = P^A_{kj}Q_{Ai}{}^h,$$
$$(4.1) \qquad R_{kji}{}^h = W_kS_{ji}{}^h - W_jS_{ki}{}^h,$$

for R on U, we have

$$W_kS_{ji}{}^h - W_jS_{ki}{}^h = P^A_{kj}Q_{Ai}{}^h.$$

Since W has no zero in U, there exists on U some vector X^k such that $W_kX^k=1$. Contraction of the above equation by X^k gives

$$S_{ji}{}^h = (P^A_{kj}X^k)Q_{Ai}{}^h + W_j(S_{ki}{}^hX^k),$$

which we write as

$$(4.6) \qquad S_{ji}{}^h = W^A_jQ_{Ai}{}^h + W_jQ_i{}^h.$$

Using this in (4.1), we obtain

$$(4.7) \qquad R_{kji}{}^h = (W_kW^A_j - W_jW^A_k)Q_{Ai}{}^h.$$

The r tensors $Q_{Ai}{}^h$ being independent, comparison of (2.2) with (4.7) gives

$$(4.8) \qquad P^A_{kj} = W_kW^A_j - W_jW^A_k.$$

Next, we show that the $r+1$ covectors W_k, W^A_k are everywhere (in U) independent. Assume that this is not the case, i.e., at some point u of U, $W_k(u)$ and $W^A_k(u)$ are not independent. Then there exist some constants c and c_A, not all zero, such that

$$cW_k(u) + c_AW^A_k(u) = 0,$$

where, since $W(u)\neq 0$, the c_A are not all zero. On account of this, we have from (4.8) that

$$\begin{aligned} c_AP^A_{kj}(u) &= W_k(u)c_AW^A_j(u) - W_j(u)c_AW^A_k(u) \\ &= -W_k(u)cW_j(u) + W_j(u)cW_k(u) = 0. \end{aligned}$$

But this contradicts the fact the r tensors P^A_{kj} are everywhere independent. Hence the $r+1$ vectors W_k, W^A_k are everywhere independent.

Lastly, from the symmetry of $S_{ji}{}^h$ in j, i, and from (4.6) we obtain

$$W_j^A Q_{Ai}{}^h + W_j Q_i^h = W_i^A Q_{Aj}{}^h + W_i Q_j^h. \tag{4.9}$$

Since W_j, W_j^A are everywhere independent, we can complete them to a coframe W_j^α $(1 \leqq \alpha \leqq n)$. Let W_α^j be its dual frame. Then contraction of (4.9) by W_B^j gives

$$Q_{Bi}{}^h = W_i^A(W_B^j Q_{Aj}{}^h) + W_i(W_B^j Q_j^h).$$

We write this as

$$Q_{Ai}{}^h = W_i Z_A^h + W_i^B Z_{AB}^h \tag{4.10}$$

and substitute it in (4.9). The result is

$$W_i(W_j^A Z_A^h - Q_j^h) - W_j(W_i^A Z_A^h - Q_i^h) = W_j^B W_i^A (Z_{AB}^h - Z_{BA}^h). \tag{4.11}$$

Contracting this by W_1^j, we get

$$Q_i^h = W_i Z^h + W_i^A Z_A^h, \quad \text{where} \quad Z^h = W_1^j Q_j^h.$$

Consequently (4.11) reduces to $Z_{AB}^h = Z_{BA}^h$. This and (4.10) and (4.7) complete the proof of our theorem.

An easy consequence of (4.5) and Theorem 1.4(d) is

THEOREM 4.4. *The covectors and vectors W_k^A, Z_A^h, Z_{AB}^h which appear in Theorem* 4.3 *are locally defined. If U, U^* are any two intersecting coordinate neighborhoods on each of which there is a decomposition of R of the form* (4.5), *then on $U \cap U^*$, we have*

$$\begin{aligned} W^{*A^*}_{\ k} &= \phi_A^{A^*} W_k^A + \phi^{A^*} W_k, \\ Z_{A^*}^{*h} &= \phi_{A^*}^A Z_A^h - \phi^{B^*} \phi_{A^*}^A \phi_{B^*}^B Z_{AB}^h, \qquad (2 \leqq A, B, A^*, B^* \leqq r+1) \\ Z_{A^*B^*}^{*h} &= \phi_{A^*}^A \phi_{B^*}^B Z_{AB}^h \end{aligned} \tag{4.12}$$

where ϕ^{A^*}, $\phi_A^{A^*}$, $\phi_{A^*}^A$ are scalars on $U \cap U^*$ such that

$$\phi_A^{A^*} \phi_{A^*}^B = \delta_A^B.$$

We now prove

THEOREM 4.5. *The recurrence covector W and the local covectors and vectors W_k^A, Z_A^h, Z_{AB}^h which appear in*

$$R_{kji}{}^h = (W_k W_j^A - W_j W_k^A)(W_i Z_A^h + W_i^B Z_{AB}^h), \qquad (2 \leqq A, B, C \leqq r+1)$$

of Theorem 4.3 *satisfy the following relations*:

$$(4.13)\qquad r>1:\begin{cases}\nabla_l W_k = N_l W_k,\\ \nabla_l \overset{A}{W}_k = \overset{A}{N}_{lB}\overset{B}{W}_k + \overset{A}{N}_l W_k,\\ \nabla_l \overset{h}{Z}_A = \overset{C}{\tilde{L}}_{lA}\overset{h}{Z}_C - N_l \overset{h}{Z}_A - \overset{C}{N}_l \overset{h}{Z}_{AC},\\ \nabla_l \overset{h}{Z}_{AB} = \overset{C}{\tilde{L}}_{lA}\overset{h}{Z}_{CB} - \overset{C}{N}_{lB}\overset{h}{Z}_{AC}.\end{cases}$$

$$(4.14)\qquad r=1:\begin{cases}\nabla_l W_k = N_l W_k + N_{l2}\overset{2}{W}_k,\\ \nabla_l \overset{2}{W}_k = \overset{2}{N}_l W_k + \overset{2}{N}_{l2}\overset{2}{W}_k,\\ \nabla_l \overset{h}{Z}_2 = -\overset{2}{N}_l \overset{h}{Z}_{22} + (\overset{2}{\tilde{L}}_{l2} - N_l)\overset{h}{Z}_2,\\ \nabla_l \overset{h}{Z}_{22} = (\overset{2}{\tilde{L}}_{l2} - \overset{2}{N}_{l2})\overset{h}{Z}_{22} - N_{l2}\overset{h}{Z}_2.\end{cases}$$

Proof. In Theorem 4.3 we have shown that for each point u of $M\backslash M_0$, there exists some coordinate neighborhood $U\ni u$ on which

$$R_{kji}{}^h = \overset{A}{P}_{kj}Q_{Ai}{}^h \qquad (2\leqq A, B\leqq r+1),$$

where

$$(4.15)\qquad \overset{A}{P}_{kj} = W_k\overset{A}{W}_j - W_j\overset{A}{W}_k,\qquad Q_{Ai}{}^h = W_i\overset{h}{Z}_A + \overset{B}{W}_i\overset{h}{Z}_{AB}.$$

On the other hand, we have from Theorem 2.4 that

$$(4.16)\qquad \nabla_l\overset{A}{P}_{kj} = \overset{A}{L}_{lB}P_{kj},\qquad \nabla_l Q_{Bi}{}^h = \overset{A}{\tilde{L}}_{lB}Q_{Ai}{}^h.$$

Substituting (4.15) in $(4.16)_1$, we have

$$(4.17)\qquad \begin{aligned} &W_k(\nabla_l\overset{A}{W}_j) + (\nabla_l W_k)\overset{A}{W}_j - W_j(\nabla_l\overset{A}{W}_k) - (\nabla_l W_j)\overset{A}{W}_k\\ &\qquad = \overset{A}{L}_{lB}(W_k\overset{B}{W}_j - W_j\overset{B}{W}_k).\end{aligned}$$

Let us complete W_k, $\overset{A}{W}_k$ to a coframe $\overset{\alpha}{W}_k$ $(1\leqq\alpha\leqq n)$ and let $\overset{k}{W}_\alpha$ be its dual frame. Then contraction of (4.17) by $\overset{k}{W}_1$ and $\overset{k}{W}_B$ gives

$$(4.18)\qquad \begin{aligned}\nabla_l\overset{A}{W}_j &= [\overset{A}{L}_{lB} - \overset{A}{\delta}_B(\nabla_l W_k)\overset{k}{W}_1]\overset{B}{W}_j + (\nabla_l\overset{A}{W}_k)\overset{k}{W}_1 W_j,\\ (\nabla_l W_k)\overset{k}{W}_B\overset{A}{W}_j &- (\nabla_l\overset{A}{W}_k)\overset{k}{W}_B W_j - \overset{A}{\delta}_B\nabla_l W_j = -\overset{A}{L}_{lB}W_j.\end{aligned}$$

Substituting in the latter equation the expression (4.18) for $\nabla_l\overset{A}{W}_k$, we get

$$(\nabla_l W_k)\overset{k}{W}_B\overset{A}{W}_j - [\overset{A}{L}_{lB} - \overset{A}{\delta}_B(\nabla_l W_k)\overset{k}{W}_1]W_j - \overset{A}{\delta}_B\nabla_l W_j = -\overset{A}{L}_{lB}W_j,$$

i.e.,

$$(4.19)\qquad \overset{A}{\delta}_B[\nabla_l W_j - (\nabla_l W_k)\overset{k}{W}_1 W_j] = (\nabla_l W_k)\overset{k}{W}_B\overset{A}{W}_j.$$

It can be verified easily that (4.18) and (4.19) together are equivalent to (4.17) and thus to $(4.16)_1$.

Two cases arise according as $r>1$ or $r=1$.

Case 1. $r>1$. For $A\neq B$, (4.19) gives

$$(\nabla_l W_j)W_B^j W_k^A = 0.$$

For any fixed B, choose an $A\neq B$. Since the covector W^A has no zero, the above equation gives

$$(\nabla_l W_j)W_B^j = 0,$$

which is therefore true for every $B=2, \cdots, r+1$. On account of this, (4.19) reduces to

$$\nabla_l W_k = (\nabla_l W_j)W_1^j W_k. \tag{4.20}$$

We note that in this case ($r>1$), the recurrence covector W is recurrent.

Case 2. $r=1$. Then $A=B=2$, and (4.19) becomes

$$\nabla_l W_k = (\nabla_l W_j)W_1^j W_k + (\nabla_l W_j)W_2^j W_k^2. \tag{4.21}$$

Now putting

$$\begin{cases} N_l = (\nabla_l W_j)W_1^j, & N_{l2} = (\nabla_l W_j)W_2^j, \\ N_l^A = (\nabla_l W_j^A)W_1^j, & N_{lB}^A = L_{lB}^A - \delta_B^A N_l \end{cases} \tag{4.22}$$

in (4.20), (4.18) and (4.21), we obtain the first parts of (4.13) and (4.14) in the theorem.

Next, we consider equation $(4.16)_2$. For simplicity, let us put

$$2 \leqq A, B \leqq r+1, \qquad 1 \leqq \overline{A}, \overline{B}, \overline{C} \leqq r+1.$$

Then we have from $(4.15)_2$ and the first parts of (4.13) and (4.14) that

$$Q_{Ai}{}^h = W_i^{\overline{B}} Z_{A\overline{B}}^h,$$

where $Z_{AB}^h = Z_{BA}^h$, $Z_{A1}^h = Z_A^h$;

$$\nabla_l W_i^{\overline{C}} = N_{l\overline{B}}^{\overline{C}} W_i^{\overline{B}},$$

where $N_{lB}^1 = 0$ if $r>1$. On account of these, equation $(4.16)_2$, namely, $\nabla_l Q_{Ai}{}^h = \overline{L}_{lA}^B Q_{Bi}{}^h$ can be written

$$\begin{aligned} \overline{L}_{lA}^C W_i^{\overline{B}} Z_{C\overline{B}}^h = \nabla_l(W_{\cdot}^{\overline{B}} Z_{A\overline{B}}^h) &= (\nabla_l W_i^{\overline{B}})Z_{A\overline{B}}^h + W_i^{\overline{B}} \nabla_l Z_{A\overline{B}}^h \\ &= N_{l\overline{C}}^{\overline{B}} W_i^{\overline{C}} Z_{A\overline{B}}^h + W_i^{\overline{B}} \nabla_l Z_{A\overline{B}}^h, \end{aligned}$$

which is equivalent to

$$\nabla_l Z^h_{A\bar{B}} = \tilde{L}^C_{lA} Z^h_{C\bar{B}} - N^{\bar{C}}_{l\bar{B}} Z^h_{A\bar{C}}.$$

This is the condensed form of the second parts of (4.13) and (4.14). Our theorem is thus completely proved.

5. Certain parallel fields of coplanes and planes on M and on $M\backslash M_0$

The decompositions of the curvature tensor into sums of tensor products in the last section give rise to the local fields of vectors Z_A, Z_{AB} and covectors W^A in $M\backslash M_0$. Here it will be shown that they (together with the recurrence covector W) span globally several parallel fields of planes or coplanes on $M\backslash M_0$. The pseudo-orthogonality between some of these parallel fields of planes and coplanes is closely related (i) to the rank of the Ricci tensor which will be proved to be always $\leqq 2$, and, (ii) in the case when the tensor ∇W is symmetric, to the condition for the holonomy group to be abelian.

5.1. **The fields $D(W)$, $D(W, W^A)$ of coplanes and the fields $D(Z_{AB})$, $D(Z_A, Z_{AB})$ of planes.**

THEOREM 5.1. *Let M be a connected n-dimensional C^∞-manifold with a linear connexion for which $T=0$, $\nabla R = W\otimes R$, $W\neq 0$, and the holonomy group is of dimension r. Let M_0 be the subset $\{u\in M\colon W(u)=0\}$ and M_1 any connected component of $M\backslash M_0$, and let the curvature tensor be locally decomposed in $M\backslash M_0$ as*

$$R_{kji}{}^h = (W_k W^A_j - W_j W^A_k)(W_i Z^h_A + W^B_i Z^h_{AB}), \qquad (2 \leqq A, B \leqq r+1).$$

Then

(a) *On each M_1 the local fields of $(r+1)$-coplanes spanned by the covectors W_k, W^A_k piece together into a parallel field $D(W, W^A)$ of $(r+1)$-coplanes. The k-support of R is a parallel field of $(r+1)$-coplanes on M whose restriction to M_1 is $D(W, W^A)$.*

(b) *On each M_1 the local fields of planes spanned by the vectors Z^h_A, Z^h_{AB} piece together into a parallel field $D(Z_A, Z_{AB})$ of planes. The h-support of R is a parallel field of planes on M whose restriction to M_1 is $D(Z_A, Z_{AB})$.*

(c) *On each M_1, the local fields of planes spanned by the vectors Z^h_{AB} piece together into a field $D(Z_{AB})$ of planes, but $D(Z_{AB})$ is not necessarily of the same dimension everywhere in M_1.*

(d) *If the holonomy group is of dimension $r>1$, then*

(i) *$D(W)$ is a parallel field of 1-coplanes on $M\backslash M_0$.*

(ii) *On each M_1, the field $D(Z_{AB})$ of planes defined in* (c) *is a parallel field of planes everywhere pseudo-orthogonal to the parallel field $D(W)$ of 1-coplanes. (We note here that in contrast with the other parallel fields described above, the* *parallel fields $D(Z_{AB})$ on different connected components of $M\backslash M_0$ may not be of the same dimension.*)

Proof. In consequence of Theorem 2.2, to prove (a), it suffices to prove that at any point $u\in M\backslash M_0$, the k-support of R coincides with the $(r+1)$-coplane spanned by the covectors W_k, W_k^A. Now let us complete W_k, W_k^A to a coframe W_k^α and let W_α^h be its dual frame. Then the k-support of R is spanned by the n^3 covectors:

$$\begin{aligned} R_{k\alpha\mu}{}^{\lambda} &= R_{kji}{}^{h} W_\alpha^j W_\mu^i W_h^\lambda \\ &= (W_k W_j^A - W_j W_k^A) W_\alpha^j Q_{Ai}{}^{h} W_\mu^i W_h^\lambda \\ &= (W_k \delta_\alpha^A - \delta_\alpha^1 W_k^A) Q_{A\mu}{}^{\lambda}. \end{aligned}$$

Therefore, the k-support is contained in the $(r+1)$-coplane spanned by W_k, W_k^A.

To show that the k-support contains all the $r+1$ covectors W_k, W_k^A, we first consider the covectors

$$R_{k1\mu}{}^{\lambda} = - Q_{A\mu}{}^{\lambda} W_k^A.$$

The number of independent covectors among them is equal to the rank of the matrix $(Q_{A\mu}{}^{\lambda})$, where A denotes the row and (μ, λ) the column. But this rank is equal to r because the r tensors $Q_{Ai}{}^{h}$ are independent. Therefore, the k-support of R contains all the covectors W_k^A.

Next consider the covectors

$$R_{k2\mu}{}^{\lambda} = Q_{2\mu}{}^{\lambda} W_k.$$

Since the r tensors Q_{Ai}^h are independent, at least one of the components $Q_{2\mu}^\lambda$ is not zero. Therefore, the k-support of R contains the covector W_k also. This completes the proof of (a).

To prove (b), we use the same coframe W_k^α as above, and obtain

$$R_{\beta\alpha\mu}{}^{h} = (\delta_\beta^1 \delta_\mu^A - \delta_\mu^1 \delta_\beta^A)(\delta_\alpha^1 Z_A^h + \delta_\alpha^B Z_{AB}^h).$$

From this and

$$R_{1A1}{}^{h} = Z_A^h, \qquad R_{1AB}{}^{h} = Z_{AB}^h,$$

it follows that the h-support of R coincides with the plane spanned by the vectors Z_A^h and Z_{AB}^h, and this proves (b).

(c) follows at once from $(4.12)_3$.

To prove (d), we observe that (i) and the first part of (ii) are consequences of (4.12), (4.13) and Theorem 1.1. It remains to prove that the parallel field

$D(Z_{AB})$ of planes is everywhere pseudo-orthogonal to the parallel field $D(W)$ of 1-coplanes, i.e. to prove that

$$W_h Z^h_{BC} = 0 \quad \text{if} \quad r > 1. \tag{5.1}$$

For this purpose, let us substitute

$$P^A_{kj} = W_k W^A_j - W_j W^A_k, \qquad Q_{Ai}{}^h = W_i Z^h_A + W^B_i Z^h_{AB}$$

into $(2.4)_1$, namely,

$$P^A_{a[k} Q_{Bj]}{}^a = C^A_{BC} P^C_{kj}.$$

Then we have

$$\begin{aligned} C^A_{BC}(W_k W^C_j - W_j W^C_k) &= (W_a W^A_{[k} - W_{[k} W^A_a)(W_{j]} Z^a_B + W^C_{j]} Z^a_{BC}) \\ &= (W_a Z^a_B)(W^A_k W_j - W^A_j W_k) - (W^A_a Z^a_{BC})(W_k W^C_j - W_j W^C_k) \\ &\quad + (W_a Z^a_{BC})(W^A_k W^C_j - W^A_j W^C_k), \end{aligned}$$

i.e.,

$$(C^A_{BC} + W_a Z^a_B \delta^A_C + W^A_a Z^a_{BC})(W_k W^C_j - W_j W^C_k) = (W_a Z^a_{BC})(W^A_k W^C_j - W^A_j W^C_k).$$

Comparison of the coefficient of W_k shows that the left side is zero, and so also is the right side. From this it follows that $W_a Z^a_{BC} = 0$ if $r > 1$, as was to be proved. Note that we have also proved that

$$C^A_{BC} + W_a Z^a_B \delta^A_C + W^A_a Z^a_{BC} = 0 \quad \text{if} \quad r > 1. \tag{5.2}$$

5.2. **Rank of the Ricci tensor.**

THEOREM 5.2. *For a linear connexion with $T=0$, $\nabla R = W \otimes R$ and $W \neq 0$, the Ricci tensor is of constant rank $\leqq 2$. The Ricci tensor is of rank 0 (i.e., R_{ji} is everywhere zero) if the parallel fields $D(W, W^A)$, $D(Z_A, Z_{AB})$ are pseudo-orthogonal; it is of rank $\leqq 1$ if the field $D(W)$ of 1-coplanes and the parallel field of planes $D(Z_A, Z_{AB})$ are pseudo-orthogonal, or if the parallel field $D(W, W^A)$ of $(r+1)$-coplanes and the field $D(Z_{AB})$ of planes are pseudo-orthogonal.*

Proof. We have proved in Theorem 2.1 that the rank of R_{ji} is constant on M. Now we deduce from the local decomposition (4.5) of R that

$$\begin{aligned} R_{ji} = R_{hji}{}^h = (W_h Z^h_{AB}) W^A_j W^B_i + (W_h Z^h_A) W^A_j W_i \\ - (W^A_h Z^h_{AB}) W_j W^B_i - (W^A_h Z^h_A) W_j W_i. \end{aligned}$$

It follows from this that the rank of R_{ji} is equal to the rank of the $(r+1) \times (r+1)$ matrix

$$\begin{bmatrix} W_h Z^h_{AB} & W_h Z^h_A \\ -W^A_h Z^h_{AB} & -W^A_h Z^h_A \end{bmatrix}. \tag{5.3}$$

If $r=1$, then R_{ji} is obviously of rank $\leqq 2$. If $r>1$, then by (5.1) $W_h Z^h_{AB}=0$. Therefore, it is seen from (5.3) that in this case also, R_{ji} is of rank $\leqq 2$. The remaining part of this theorem is an immediate consequence of (5.3).

5.3. **A theorem on the holonomy group when ∇W is symmetric.** It is known (Walker [8, §6]) that for a Riemannian connexion with $\nabla R = W\otimes R$ and $W\neq 0$, the tensor ∇W is symmetric. On the other hand Hlavaty [2, Theorem 3.3] has recently proved that the (local) holonomy group of such a Riemannian connexion is abelian. We now prove

THEOREM 5.3. *For a linear connexion with $T=0$, $\nabla R = W\otimes R$, $W\neq 0$ and ∇W symmetric, the holonomy group H is abelian iff*

(a) dim $H=1$, *or*

(b) dim $H>1$, *the parallel fields $D(W)$ and $D(Z_A, Z_{AB})$ are pseudo-orthogonal, and the parallel fields $D(W, W^A)$ and $D(Z_{AB})$ are pseudo-orthogonal.*

Proof. If dim $H=r=1$, H is of course abelian.

Let us assume then dim $H=r>1$. We have from (2.4) that

$$Q_{[Ba}{}^{h} Q_{C]i}{}^{a} = \tilde{C}^A_{BC} Q_{Ai}{}^{h}, \tag{5.4}$$

$$(\nabla_{[m} W_{l]})\delta^A_C = (C^A_{BC} + \tilde{C}^A_{BC}) P^B_{ml}, \tag{5.5}$$

where $2\leqq A, B, C\leqq r+1$. Equation (5.4) shows that the condition for H to be abelian is that $\tilde{C}^A_{BC}=0$ (see corollary to Theorem 2.4). Since ∇W is symmetric and the r tensors P^C_{ml} are independent, it follows from (5.5) that $\tilde{C}^A_{BC} = -C^A_{BC}$. Therefore, the condition for H to be abelian is $C^A_{BC}=0$, i.e., by (5.2)

$$W_a Z^a_B \delta^A_C + W^A_a Z^a_{BC} = 0, \qquad (2\leqq A, B, C \leqq r+1; r>1). \tag{5.6}$$

We now show that (5.6) is equivalent to

$$W_a Z^a_B = 0, \qquad W^A_a Z^a_{BC} = 0. \tag{5.7}$$

Since $Z^a_{BC} = Z^a_{CB}$, equation (5.6) implies that

$$W_a Z^a_B \delta^A_C = W_a Z^a_C \delta^A_B.$$

If in this equation we let B be arbitrary but fixed and $A=C\neq B$, we obtain $W_a Z^a_B=0$. Therefore, $(5.7)_1$ is true and, on account of this, (5.6) reduces to $(5.7)_2$. Our theorem now follows from (5.7) and (5.1): $W_h Z^h_{BC}=0$ which holds for $r>1$.

6. Decompositions of the tensor ∇W

In Theorem 4.5 we proved that for each point u of M at which $W(u)\neq 0$, there exists a coordinate neighborhood $U\ni u$ on which ∇W can be decomposed into

$$\nabla_l W_k = N_l W_k \qquad \text{if } r > 1,$$

or

$$\nabla_l W_k = N_l W_k + \tilde{N}_l \tilde{W}_k \qquad \text{if } r = 1,$$

where r is the dimension of the holonomy group, and N_l, $\tilde{N}_l(=N_{l2})$, $\tilde{W}_k(=W_k^2)$ are covectors on U.

Since both the recurrence covector W and its covariant derivative ∇W are globally defined on M and therefore on $M\backslash M_0$, we naturally want to know whether there exist global covectors N, $\tilde{N}$, $\tilde{W}$ on $M\backslash M_0$ so that the above decomposition of ∇W holds globally on $M\backslash M_0$. The main purpose of this section is to study this problem.

6.1. **The case** dim $H>1$.

Theorem 6.1. *Let M be a connected n-dimensional C^∞-manifold with a linear connexion for which $T=0$, $\nabla R=W\otimes R$ and $W\not\equiv 0$, and let M_0 be the subset $\{u\in M: W(u)=0\}$. If the holonomy group is of dimension $r>1$, then either*

(a) *$\nabla W=\rho W\otimes W$ everywhere on $M\backslash M_0$, or*

(b) *$\nabla W=N\otimes W$ and $\nabla N=(W-N)\otimes N+V\otimes W$ everywhere on $M\backslash M_0$, where the covectors N and V are uniquely determined and N is nowhere dependent on W. It follows that in case* (b), *the covectors W and N span a parallel field $D(W, N)$ of 2-coplanes on $M\backslash M_0$.*

Proof. Let $r>1$. We have proved in Theorem 4.5 that

$$\nabla_l W_k = N_l W_k \qquad \text{locally in } M\backslash M_0. \tag{6.1}$$

Since the tensor $\nabla_{[l}W_{k]}$ on M is either everywhere zero or nowhere zero in M (cf. Theorem 3.1) and by (6.1), $\nabla_{[l}W_{k]}=N_{[l}W_{k]}$, we may conclude that the local covector N_l in $M\backslash M_0$ is either everywhere dependent on W_l or nowhere dependent on W_l.

In the first case we have $\nabla_l W_k=\rho W_l W_k$, and (a) is proved.

Consider further the second case where N_l is nowhere dependent on W_l. The covector N_l is uniquely determined by (6.1). For, if N_l^* is another covector such that $\nabla_l W_k=N_l^* W_k$, then from $N_l^* W_k=\nabla_l W_k=N_l W_k$, we obtain $(N_l^*-N_l)W_k=0$. Since W is nowhere zero in $M\backslash M_0$, this is equivalent to $N_l^*=N_l$. In other words, we have a unique covector N on $M\backslash M_0$ such that $\nabla W=N\otimes W$ everywhere in $M\backslash M_0$.

Substituting $\nabla_l W_k=N_l W_k$ in $\nabla_m(\nabla_{[l}W_{k]})=W_m\nabla_{[l}W_{k]}$ (cf. Theorem 3.1), we obtain

$$W_m N_{[l} W_{k]} = \nabla_m (N_{[l} W_{k]}) = (\nabla_m N_{[l}) W_{k]} + N_{[l} \nabla_m W_{k]}$$
$$= (\nabla_m N_{[l}) W_{k]} + N_{[l} N_m W_{k]}.$$

Therefore,

$$[\nabla_m N_{[l} - (W_m - N_m) N_{[l}] W_{k]} = 0.$$

Hence

$$\nabla_m N_l = (W_m - N_m) N_l + V_m W_l,$$

where V_m is some covector on U. It follows from this (by the argument used above on N_l) that there exists on $M \backslash M_0$ a unique covector V such that

$$\nabla N = (W - N) \otimes N + V \otimes W.$$

Combining this equation with $\nabla W = N \otimes W$ and the fact that N is nowhere dependent on W, we see (cf. Theorem 1.1) that the covectors W, N span a parallel field of 2-coplanes on $M \backslash M_0$. This completes the proof of the theorem.

6.2. **The case** $\dim H = 1$. In this case, the tensors R and ∇W can be locally decomposed in $M \backslash M_0$ as

$$\begin{aligned} R_{kji}{}^h &= (W_k \tilde{W}_j - W_j \tilde{W}_k)(W_i Z^h + \tilde{W}_i \tilde{Z}^h), \\ \nabla_l W_k &= N_l W_k + \tilde{N}_l \tilde{W}_k. \end{aligned} \tag{6.2}$$

Furthermore, the field $D(W, \tilde{W})$ of 2-coplanes on $M \backslash M_0$ is the restriction to $M \backslash M_0$ of the k-support of R. We now prove

THEOREM 6.2. *If* $\dim H = 1$, *the local covectors* N_l, $\tilde{N}_l$ *in* $M \backslash M_0$ *which appear in* (6.2) *have the following properties*:

(a) $\tilde{N}_l$ *is everywhere lying in the k-support of R.*

(b) N_l *is either everywhere or nowhere lying in the k-support of R.*

(c) *If* N_l *is nowhere lying in the k-support of R, then* $D(N, W, \tilde{W})$ *is a parallel field of 3-coplanes on* $M \backslash M_0$.

Proof. Substituting (6.2) in (3.2), namely, $\operatorname{cyc}(W_m \nabla_{[l} W_{k]}) = 0$, we have

$$\operatorname{cyc} W_m (N_{[l} W_{k]} + \tilde{N}_{[l} \tilde{W}_{k]}) = 0, \qquad \text{i.e.} \quad \operatorname{cyc}(W_m \tilde{N}_{[l} \tilde{W}_{k]}) = 0.$$

From this, since W_l and $\tilde{W}_l$ are everywhere independent, it follows that

$$\tilde{N}_l = \rho W_l + \tilde{\rho} \tilde{W}_l, \tag{6.3}$$

which proves (a).

On account of (6.3), we have

$$\nabla_{[l} W_{k]} = N_{[l} W_{k]} + \tilde{N}_{[l} \tilde{W}_{k]} = N_{[l} W_{k]} - \rho \tilde{W}_{[l} W_{k]} = (N_{[l} - \rho \tilde{W}_{[l}) W_{k]},$$

i.e.

$$\nabla_{[l} W_{k]} = \overline{N}_{[l} W_{k]}, \quad \text{where} \quad \overline{N}_l = N_l - \rho \tilde{W}_l. \tag{6.4}$$

Combining this with (3.1), namely,

$$\nabla_m(\nabla_{[l}W_{k]}) = W_m\nabla_{[l}W_{k]}, \tag{6.5}$$

we see that $\overline{N}_{[l}W_{k]}$ is everywhere zero or nowhere zero; in other words, either N_l is everywhere dependent on W_l and $\tilde{W}_l$ or nowhere dependent on W_l and $\tilde{W}_l$. This proves (b).

Consider further the case where $\overline{N}_{[l}W_{k]}$ is nowhere zero. Substituting (6.4) in (6.5), we get

$$W_m\overline{N}_{[l}W_{k]} = \nabla_m(\overline{N}_{[l}W_{k]}) = (\nabla_m\overline{N}_{[l})W_{k]} + \overline{N}_{[l}(N_mW_{k]} + \tilde{N}_m\tilde{W}_{k]}),$$

i.e.

$$[\nabla_m\overline{N}_{[l} - (W_m - N_m)\overline{N}_{[l}]W_{k]} + \tilde{N}_m\overline{N}_{[l}\tilde{W}_{k]} = 0.$$

Applying Cartan's lemma to this equation, we see that $\nabla_m\overline{N}_l$ is of the form

$$\nabla_m\overline{N}_l = (W_m - N_m)\overline{N}_l + A_mW_l + \tilde{A}_m\tilde{W}_l.$$

If we substitute $\overline{N}_l = N_l - \rho\tilde{W}_l$ in this, and take into account that $\nabla_m\tilde{W}_l = N_m^2W_l + N_{m2}^2\tilde{W}_l$ (cf. (4.14)), the result is of the form

$$\nabla_mN_l = B_mN_l + C_mW_l + \tilde{C}_m\tilde{W}_l. \tag{6.6}$$

On account of this and Theorem 1.1, (c) will be completely proved if we let U, U^* be coordinate neighborhoods in $M\backslash M_0$ on which (6.2) and $\nabla_lW_k = N_l^*W_k + \tilde{N}_l^*\tilde{W}_k^*$ respectively hold and are able to show that at every point of $U\cap U^*$, the covectors W, $\tilde{W}^*$, N^* and the covectors W, $\tilde{W}$, N span the same coplane. We already know that $\tilde{W}_k^*$ is linearly dependent on W_k and $\tilde{W}_k$ (Theorem 4.4). In addition,

$$\overset{*}{N}_lW_k + \overset{*}{\tilde{N}}_l\overset{*}{\tilde{W}}_k = \nabla_lW_k = N_lW_k + \tilde{N}_l\tilde{W}_k$$

hold on $U\cap U^*$. Contracting this by a vector X^k such that $W_kX^k = 1$ and $\tilde{W}_k^*X^k = 0$, we see that N_l^* is linearly dependent on N_l and $\tilde{N}_l$, and consequently, by (a), on N_l, W_l and $\tilde{W}_l$. This completes the proof of the theorem.

The following theorem gives a partial answer to the question whether there exist global covectors N, $\tilde{N}$, $\tilde{W}$ on $M\backslash M_0$ such that equation (6.2) holds globally on $M\backslash M_0$.

THEOREM 6.3. *Let* dim $H = 1$, *and let* M_{00} *be the subset*

$$\{u \in M: [(\nabla_lW_{[k})W_{j]}](u) = 0\}$$

of $M\backslash M_0$. *Then*

(a) *On* M_{00}, *there exists a unique covector* N *such that*

$$\nabla W = N \otimes W.$$

(b) *On* $(M\backslash M_0)\backslash M_{00}$, *there exist*

(i) *a covector N which is an extension of the covector N on M_{00} defined in* (a), *and*

(ii) *a tensor E of type* (0, 2) *and rank* 1

such that
$$\nabla W = N \otimes W + E.$$

(c) *If E_{lk} are the components of E in any local coordinate system in $(M\backslash M_0)\backslash M_{00}$, then at every point of $(M\backslash M_0)\backslash M_{00}$,*

(i) *both the k-support and the l-support of E lie in the k-support $(=D(W, \tilde{W}))$ of the curvature tensor R, and*

(ii) *the k-support of E nowhere contains the recurrence covector W.*

Proof. It follows from (6.2) that on a suitable coordinate neighborhood U of each point in $M\backslash M_0$,

$$(\nabla_l W_{[k})W_{j]} = \tilde{N}_l \tilde{W}_{[k}W_{j]}.$$

Since $\tilde{W}_{[k}W_{j]}$ has no zero in U, the tensor $(\nabla_l W_{[k})W_{j]}$ and the covector $\tilde{N}_l$ have common zeros in U. But $(\nabla_l W_{[k})W_{j]}$ is a tensor on $M\backslash M_0$. Therefore, if M_{00} is defined as in the theorem, we have $\nabla_l W_k = N_l W_k$ locally in M_{00}. Consequently (by an argument used in §6.1), there exists on M_{00} a unique covector N such that $\nabla W = N \otimes W$. This proves (a).

Let us now consider the situation on the submanifold $(M\backslash M_0)\backslash M_{00}$ on which $\nabla_l W_k$ can be locally expressed as

$$\nabla_l W_k = N_l W_k + \tilde{N}_l \tilde{W}_k, \tag{6.2}$$

where $\tilde{N}_l$ has no zero. We recall that the vector $\tilde{W}_k$ was first introduced in the second local decomposition (§4.3) of the curvature tensor R in $M\backslash M_0$. Let U be a coordinate neighborhood in $(M\backslash M_0)\backslash M_{00}$ on which R can be decomposed as

$$R_{kji}{}^h = (W_k \tilde{W}_j - W_j \tilde{W}_k)(W_i Z^h + \tilde{W}_i \tilde{Z}^h),$$

where $\tilde{W}_k$ is everywhere independent of W_k. By Theorem 4.4, the most general covector $\tilde{W}^*_k$ which can appear in the second decomposition of R is of the form

$$\tilde{W}^*_k = \tilde{\phi}\tilde{W}_k + \phi W_k, \quad \text{where} \quad \tilde{\phi} \neq 0.$$

If at a point $u \in (M\backslash M_0)\backslash M_{00}$, $\nabla_l W_k$ can be expressed also as

$$\nabla_l W_k = \overset{*}{N}_l W_k + \tilde{\overset{*}{N}}_l \tilde{W}^*_k, \tag{6.2*}$$

then

$$\begin{aligned} N_l W_k + \tilde{N}_l \tilde{W}_k &= \overset{*}{N}_l W_k + \tilde{\overset{*}{N}}_l \tilde{W}^*_k \\ &= \overset{*}{N}_l W_k + \tilde{\overset{*}{N}}_l(\tilde{\phi}\tilde{W}_k + \phi W_k) \\ &= (\overset{*}{N}_l + \phi \tilde{\overset{*}{N}}_l) W_k + \tilde{\phi}\tilde{\overset{*}{N}}_l \tilde{W}_k. \end{aligned}$$

From this it follows that

$$\tilde{\phi}\tilde{N}^*_l = \tilde{N}_l, \qquad \tilde{\phi} \neq 0, \tag{6.7}$$
$$N^*_l + \phi\tilde{N}^*_l = N_l.$$

Therefore, the most general N^*_l at u satisfying (6.2*) is

$$N^*_l = N_l - (\phi/\tilde{\phi})\tilde{N}_l. \tag{6.8}$$

Equations (6.7) and (6.8) show that on $(M\backslash M_0)\backslash M_{00}$, $\tilde{N}_l$ has a fixed direction and N^*_l depends on a single parameter varying from $-\infty$ to $+\infty$.

Consider now the bundle B^N of all covectors N_l at all points of $(M\backslash M_0)\backslash M_{00}$. It follows from the above observations that the fiber F_u over any point $u \in (M\backslash M_0)\backslash M_{00}$ is isomorphic with the real line. Therefore, by a theorem of Steenrod's on existence of cross sections on fiber bundles, C^∞ cross sections of B^N exist. Any such cross section is a vector N^* on $(M\backslash M_0)\backslash M_{00}$ having the property that in a suitable coordinate neighborhood U of each point of $(M\backslash M_0)\backslash M_{00}$,

$$\nabla_l W_k = N^*_l W_k + \tilde{N}_l \tilde{W}_k, \quad \text{i.e.} \quad \nabla_l W_k - N^*_l W_k = \tilde{N}_l \tilde{W}_k.$$

Since the left side of the last equation is the restriction to U of a tensor on $(M\backslash M_0)\backslash M_{00}$ so is also the right side $\tilde{N}_l\tilde{W}_k$. In other words, there exists on $(M\backslash M_0)\backslash M_{00}$ a tensor E of type (0, 2) and rank 1 such that $(E|\,U)_{lk} = \tilde{N}_l\tilde{W}_k$. We have thus shown that

$$\begin{aligned} \nabla W &= N \otimes W && \text{on } M_{00}, \\ \nabla W &= N^* \otimes W + E && \text{on } (M\backslash M_0)\backslash M_{00}. \end{aligned} \tag{6.9}$$

Let us now define the values of N^*, E and $\tilde{N}$ on the boundary of M_{00} by continuity. Then it is easily seen from (6.9) that

$$E = 0 \text{ (i.e. } \tilde{N} = 0) \quad \text{and} \quad N^* = N \text{ on the boundary of } M_{00}.$$

This completes the proof of (b).

Finally, (c) follows from the very definition of E and from (6.7).

6.3. **The special case where** dim $H = 1$ **and the Ricci tensor is not symmetric.** For this case, we have a better result than Theorem 6.3.

Consider first the general case where dim $H \geqq 1$. We have given the curvature tensor three different decompositions:

$$R_{kji}{}^h = W_k S_{ji}{}^h - W_j S_{ki}{}^h,$$
$$R_{kji}{}^h = \overset{A}{P}_{kj} Q_{Ai}{}^h,$$
$$R_{kji}{}^h = (W_k \overset{A}{W}_j - W_j \overset{A}{W}_k)(W_i \overset{h}{Z}_A + \overset{B}{W}_i \overset{h}{Z}_{AB}), \qquad (2 \leqq A, B \leqq r+1).$$

Of these, the first is global (on $M\backslash M_0$) while the other two are local (i.e. they hold on a suitable coordinate neighborhood U of each point in $M\backslash M_0$). By (4.6), the global tensor $S_{ji}{}^h$ can be expressed on U as

$$S_{ji}{}^{h} = W_j^A Q_{Ai}{}^{h} + W_j Q_i^h,$$

from which we have

$$S_{jh}{}^{h} = W_j^A Q_{Ah}{}^{h} + W_j Q_h^h. \tag{6.10}$$

On the other hand, it follows from the equation

$$-R_{[kj]} = R_{kjh}{}^{h} = W_k S_{jh}{}^{h} - W_j S_{kh}{}^{h}$$

that if the Ricci tensor is not symmetric, the global covector $S_{jh}{}^{h}$ is everywhere (in $M \backslash M_0$) independent of W_j. Consequently the r scalars $Q_{Ah}{}^{h}$ on U which appear in (6.10) have no common zero. Hence, by (4.12) and (6.10), we may replace one of the r covectors W_j^A on U, say W_j^2, by the restriction of $S_{jh}{}^{h}$ to U. Let us assume that this has been done. Then we have

$$W_j^2 = W_j^A Q_{Ah}{}^{h} + W_j Q_h{}^{h}, \quad \text{i.e.} \quad Q_{Ah}{}^{h} = \delta_A^2, \quad Q_h{}^{h} = 0.$$

Hence

LEMMA 6.1. *If the Ricci tensor is not symmetric, we may take as W_j^2 the restriction of $S_{jh}{}^{h}$ to U so that in the decomposition*

$$R_{kji}{}^{h} = (W_k W_j^A - W_j W_k^A)(W_i Z_A^h + W_i^B Z_{AB}^h), \qquad (2 \leqq A,\, B \leqq r+1),$$

the covector W_k^2, like W_k, is globally defined on $M \backslash M_0$, and $W_h Z_A^h + W_h^B Z_{AB}^h = \delta_A^2$.

Applying this lemma to the case where $\dim H = 1$ and using some of the results in Theorems 5.1 and 6.2, we have

THEOREM 6.4. *Let a linear connexion on M be such that $T=0$, $\nabla R = W \otimes R$ and $W \neq 0$, and let $M_0 = \{u \in M \colon W(u) = 0\}$. If the holonomy group is of dimension 1 and the Ricci tensor is not symmetric, then the curvature tensor R and the tensor ∇W can be globally decomposed on $M \backslash M_0$ as*

$$R_{kji}{}^{h} = (W_k \tilde{W}_j - W_j \tilde{W}_k)(W_i Z^h + \tilde{W}_i \tilde{Z}^h) \quad \text{with} \quad W_h Z^h + \tilde{W}_h \tilde{Z}^h = 1,$$
$$\nabla W = N \otimes W + \tilde{N} \otimes \tilde{W},$$

where all the covectors and vectors are globally defined on $M \backslash M_0$ such that

(a) *$D(\tilde{N}, W, \tilde{W}) = D(W, \tilde{W})$ is a parallel field of 2-coplanes on $M \backslash M_0$,*

(b) *$D(N, W, \tilde{W})$ is either identical with $D(W, \tilde{W})$ or is a parallel field of 3-coplanes on $M \backslash M_0$,*

(c) *$D(Z, \tilde{Z})$ is a parallel field of 1-planes or a parallel field of 2-planes on $M \backslash M_0$.*

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